Well-posedness and qualitative properties for abstract time-difference equations.

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Dedicated to
Sofía Díaz

Abstract

In this thesis we introduce the notions of the stable Lévy process and the scaled Wright function within the discrete setting. Using these notions, we prove a subordination principle which will be used to investigate different classes of discrete-time fractional difference equations. In addition, we introduce the Banach space of (N, λ)-periodic vector-valued sequences. Moreover, we show the existence and uniqueness of (N, λ)-periodic solutions to a class of abstract Volterra difference equations as well as of fractional difference equations.

Keywords. Difference equations; Fractional difference equations; Volterra difference equations; Subordination principle; Stable Lévy process; Scaled Wright function; (N, λ)-periodic discrete function.

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$\mathbb{N}$	the set of natural numbers
$\mathbb{N}_a$	the set $\{a, a+1, a+2, \cdots\}$
$\mathbb{Z}$	the set of integer numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+$	the set of non-negative real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{R}^d$	the space of d-dimensional real
$\mathbb{D}(a, r)$	the open disk of center $a \in \mathbb{C}$ and radius $r > 0$
X	a complex Banach space equipped with the norm $\|\cdot\|_X$
$\mathcal{B}(X)$	the Banach space of all bounded operators defined on X
$s(A; B)$	the vector space of all vector-valued sequence $f : A \to B$
$\ell^1(\mathbb{N}_0, X)$	the space of all sequences $f \in s(\mathbb{N}_0; X)$ such that $\displaystyle\|f\|_{\ell^1} := \sum_{n=0}^{\infty} \|f(n)\|_X < \infty$
$\Theta^1_\alpha(\mathbb{Z}, X)$	the vector space of all$f \in s(\mathbb{Z}; X)$ such that $\displaystyle\|f\|_{\Theta^1_\alpha} := \sum_{n=-\infty}^{\infty} \|n^{\alpha-1} f(n)\|_X < \infty$
$L^p(\mathbb{R}^d)$	the L^p spaces on $\mathbb{R}^d$
$\|\cdot\|_p$	the L^p-norm $\|f\|_p := \left(\int_{\mathbb{R}^d} |f(x)|^p \, dx\right)^{1/p}$ where $1 \le p < \infty$
$\|\cdot\|_\infty$	the L^∞-norm $\|f\|_\infty := \sup\{|f(x)|\}$
$\mathbb{P}_{N\lambda}(\mathbb{Z}, X)$	the set of all sequence with λ-period N
$\|\cdot\|_{N\lambda}$	$N\lambda$-norm $\|f\|_{N\lambda} := \max_{n \in [0,N] \cap \mathbb{Z}} \|\lambda\hat{}(-n) f(n)\|_X$
$\|\cdot\|_N$	N-norm $\|f\|_N := \max_{n \in [0,N] \cap \mathbb{Z}} \|f(n)\|_X$

Δ	the forward difference operator
∇	the backward difference operator
$\nabla^{-\alpha}$	the α-th fractional backward sum
${}_{RL}\nabla^{\alpha}$	the α-th Riemann-Liouville fractional backward difference
${}_{C}\nabla^{-\alpha}$	The α-th Caputo fractional backward difference
$\Delta^{-\alpha}$	the α-th fractional forward sum
${}_{RL}\Delta^{\alpha}$	the α-th Riemann-Liouville fractional forward difference
${}_{C}\Delta^{-\alpha}$	the α-th Caputo fractional forward difference
$\Delta_W^{-\alpha}$	the α-th fractional sum in the Weyl-like sense
Δ_W^{α}	the α-th fractional difference in the Weyl-like sense
$\Delta^{\alpha,\beta}$	the Hilfer fractional difference operator
$*$	the discrete convolution of two sequences $f, g \in s(\mathbb{N}_0, X)$ is defined by $(f * g)(n) := \sum_{j=0}^{n} f(n-j)g(j)$
$\star$	the discrete convolution of two sequences $f, g \in s(\mathbb{Z}, X)$ is defined by $(f \star g)(n) := \sum_{j=-\infty}^{n} f(n-j)g(j)$
$\boldsymbol{\nabla}_x$	Gradient operator
$\boldsymbol{\Delta}_x$	Laplacian operator
$\circledast$	the classical convolution of two functions $f, g \in \mathbb{R}^d$ is defined by $(f \circledast g)(x) := \displaystyle\int_{\mathbb{R}^d} f(x-y)g(y)dy$

Introduction

The theory of fractional difference equations has gained importance in recent years as it has made possible better describe different phenomena that present a discrete in time evolution. We refer to the references [50] for applications to a class of specific systems, [8] for applications to chaotic systems with short memory and image encryption, [49] for applications to fractional discrete-time neural networks, [76] for applications to image enhancement and [77] for applications to Lyapunov functions for fractional difference equations. See also the works [19, 30, 39, 48, 68, 75, 79], to mention only an excerpt from the long list of relevant publications. Nowadays, the theory is being developed in two main areas of discussion: scalar-valued and operator-valued setting. Whereas the scalar-valued setting is relatively older, the vector-valued theory began to be discussed only recently in [56] and later by other authors; see e.g. [3, 5, 6, 45] where the authors studied aspects as maximal regularity, stability and fractional discrete resolvent operators, among others.

In the vector-valued side, there are problems which have not been considered in discrete time-fractional order yet. For instance, there is great interest in fractional problems and its asymptotic behavior [4, 25, 51, 80]. Whereas in continuous time, there is an amount of papers which deals with this issue, see for example [27, 28, 53] and references therein, the problem of study the large-time behavior of solutions in discrete time for a fractional version of the d-dimensional heat equation, remains open.

Concerning methods, the concept of discrete α-resolvent sequence $\{S_{\alpha,\alpha}(n)\}_{n\in\mathbb{N}_0}$ is an important tool to have an explicit representation of the solution for the Cauchy problem in discrete-time

$$ {}_{RL}\Delta^{\alpha}u(n) = Au(n+1), \quad u(0) = u_0, \quad n \in \mathbb{N}, \tag{1} $$

where A is a closed linear operator defined in a Banach space X, $0 < \alpha \leq 1$ and ${}_{RL}\Delta^{\alpha}$ denotes the fractional forward difference operator in Riemann-Liouville like sense (see [45, 55]). In [5], the authors proved that one of the main properties of α-resolvent sequences associated to equations of type (1) is that their definition implies that $1 \in \rho(A)$, the resolvent set of A, and that there exists a scalar sequence $\{\beta_{\alpha,n}(j)\}_{n,j\in\mathbb{N}}$ such that

$$S_{\alpha,\alpha}(n)x = \sum_{j=1}^{n} \beta_{\alpha,n}(j)(I-A)^{-(j+1)}x, \quad x \in X, \quad n \in \mathbb{N}. \tag{2}$$

This result is very important for the theory because subordinates the solution of the problem (1) in case $0 < \alpha < 1$ to the solution of the problem (1) in the simple case $\alpha = 1$, namely

$$\mathcal{T}(n) := (I-A)^{-(n+1)}, \quad n \in \mathbb{N}_0. \tag{3}$$

Nonetheless, a closed and precise description of the scalar sequence $\{\beta_{\alpha,n}(j)\}_{n,j\in\mathbb{N}}$, was not given in [5] and was left as an open problem.

In more general situations, it is well known that the Riemann-Liouville and Caputo operators in continuous time are particular cases of the (continuous) Hilfer operator [46]. The study of Cauchy problems which involves the Hilfer operator have as a main ingredient resolvent families [43, 62]. However, in the case of the discrete Hilfer operator, not much seems to be known about discrete Cauchy problems and discrete resolvent sequences of operators.

On the other hand, a fundamental aspect in the qualitative study of the solutions of evolution equations in discrete time is their periodicity or anti-periodicity, among other issues.

In the papers [11,35,52], the authors analyzed several scalar discrete-time periodic systems and got periodic solutions of second kind. Periodic functions of second kind were introduced and studied by G. Floquet in [38]. The Floquet theory is important to study different mathematical models, for example the predator-prey model (see [22]). For additional related applications, see [26, 32] and the references therein. Although several authors have worked with this type of real-valued sequences (periodic of second kind), so far none has mentioned the vector-valued case.

As a first model example, we note that in the last years Volterra difference equations have been considered in several applied fields and nowadays there is a wide interest in developing the qualitative theory for such equations [10,12,13,24,31,73]. In particular, in the reference [12], the authors considered nonlinear Volterra difference equations of convolution type on a Banach space X, namely

$$u(n+1) = \sigma \sum_{j=-\infty}^{n} a(n-j)u(j) + f(n, u(n)), \quad n \in \mathbb{Z}, \quad \sigma \in \mathbb{C}, \tag{4}$$

where the kernel a and the nonlinear term f satisfy suitable conditions. It should be noted that the study of vector-valued periodic solution of second kind for Volterra difference equations (4) does not exist at this time and deserves to be investigated.

As a second model example, we observe that the existence of solutions for the abstract fractional difference equation

$$\Delta_W^{\alpha} u(n) = Au(n+1) + f(n, u(n)), \quad n \in \mathbb{Z}, \tag{5}$$

where $0 < \alpha \leq 1$, Δ_W^{α} denotes the fractional difference operator in the Weyl-like sense, A is a closed linear operator defined on a Banach space X and f is given, began to be studied in the articles [56] and [45] in its linearized form. Subsequently, maximal regularity in Lebesgue spaces of sequences was studied in [59]. In case A is bounded, weighted bounded solutions were studied in [60]. In [5], the existence of almost automorphic mild solutions was studied. However, the existence and uniqueness of second kind periodic sequence solutions for (5) is still an open problem.

The overall purpose of this thesis is to study conditions to guarantee the existence, uniqueness and qualitative properties of solutions for the above mentioned classes of models in discrete time. Thus, motivated by what is described above, we will consider the following four specific problems:

In the first problem, we investigate the large-time behavior of solutions for the following discrete in time fractional diffusion equation

$$\begin{aligned} {}_C\Delta^\alpha u(n,x) &= \mathbf{\Delta}_x u(n,x), \quad n \in \mathbb{N},\, x \in \mathbb{R}^d, \\ u(0,x) &= f(x), \end{aligned} \tag{6}$$

where $0 < \alpha \leq 1$, ${}_C\Delta^\alpha$ denotes the Caputo fractional backward difference operator, $\mathbf{\Delta}_x$ denotes the Laplace operator acting in space, u is defined on $\mathbb{N}_0 \times \mathbb{R}^d$ and f is a function defined on $\mathbb{R}^d$.

We would like to address the following questions: Is there an explicit representation of the fundamental solution of the equation (6)? Are there conditions for the asymptotic decay of solutions?

In *our second task* we would like to answer the following open problem: Is it possible to give an accurate description of the scalar sequence $\{\beta_{\alpha,n}(j)\}_{n,j\in\mathbb{N}}$? In particular, can we use families of operators to obtain a subordination principle in the discrete case for the Cauchy problem involving the Hilfer difference operator?

The third problem, that we address in this thesis is the following: Can we find conditions that guarantee the existence and uniqueness of (N,λ)-periodic solutions for the class of Volterra difference equations (4)?

Finally, in *the fourth problem*, we ask: Can we find (N,λ)-periodic solutions for the abstract fractional difference equation (5)?

In order to address the above-mentioned issues, we will proceed in the following way.

We introduce initially a new notion of Lévy α-stable distribution and scaled Wright function on discrete time (see Chapter 2) as follows:

The Lévy α–stable distribution on discrete time is defined as

$$l_\alpha(n,j) = \sum_{i=0}^{j} \binom{j}{i} (-1)^i k^{-\alpha i}(n), \quad 0 < \alpha \leq 1, \quad n \in \mathbb{N}_0, \quad j \in \mathbb{N}_0,$$

where k^γ is a sequence initially defined as

$$k^\gamma(n) := \begin{cases} \dfrac{\gamma(\gamma+1)\cdot\dots\cdot(\gamma+n-1)}{n!}, & n \in \mathbb{N}, \\ \\ 1, & n = 0. \end{cases}$$

We will prove that the discrete Lévy α–stable distribution satisfies two important properties:

$$0 \leq l_\alpha(n,j) \leq 1 \quad \text{and} \quad \sum_{i=0}^{\infty} l_\alpha(i,j) = 1.$$

See Section 2.1 below. With this notion of Lévy α–stable distribution on discrete time, we finally find that

$$\beta_{\alpha,n}(j) = l_\alpha(n,j), \quad n,j \in \mathbb{N}.$$

This solves the first part of the second open problem proposed in this thesis.

Next, for $0 \leq \beta$ and $0 < \alpha \leq 1$, we introduce *the discrete scaled Wright function* $\varphi^h_{\alpha,\beta}$ (see Section 2.1) given by

$$\varphi^h_{\alpha,\beta}(n,j) := \frac{1}{2\pi i} \int_\Upsilon \frac{1}{z^{n+1}} \frac{\left(1 - h(\frac{1-z}{h})^\alpha\right)^j}{(\frac{1-z}{h})^\beta} \, dz, \quad j,n \in \mathbb{N}_0, \quad h > 0,$$

with $\varphi^1_{\alpha,\beta}(n,j) \equiv \varphi_{\alpha,\beta}(n,j)$, and we show that $\varphi_{\alpha,0}(n,j) \equiv l_\alpha(n,j)$. Further, we define the concept of (α,ν)*-resolvent sequence* (see Section 3.1) that includes the notion of α-resolvent sequence when $\alpha = \nu$.

The first problem will be then solved in the following way. Firstly, we use the discrete scaled Wright function to write the solution of (6) as follows

$$u(n,x)=\sum_{j=1}^{\infty}\varphi_{\alpha,1-\alpha}(n-1,j-1)(\mathcal{G}_j * f)(x),\ n\in\mathbb{N}, \tag{7}$$

where $\mathcal{G}_j$ is the discrete Gaussian kernel associated to the discrete in time heat problem given in [1]. A relevant fact of (7) is that we get an equivalent representation using the Gaussian kernel or the Fox H-function as Proposition 2.2.4 and Proposition 2.2.5 show. Moreover, we state basic properties of the fundamental solution of (6). One of them is that the integral over $\mathbb{R}^d$ of the fundamental solution is 1, which gives directly the mass conservation principle for solutions (7).

Finally, the $L^p(\mathbb{R}^d)$-decay for the fundamental solution and its gradient allow to get the $L^p(\mathbb{R}^d)$-decay for the solution (7) (see Theorem 2.3.3), and also the large time behaviour. Here, the properties of the discrete scale Wright function (see Section 2.1) will be very useful.

The second problem is solved as follows. Initially, we clarify the role of the sequence of operators $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0}$ that appears in (3). In fact, we will see (in Chapter 3) that $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0}$ constitutes a *discrete C-semigroup*. This concept is defined here by the first time, but it is the analogous to the notion of C-semigroup introduced by R. deLaubenfels [33] in the 90s. The surprising fact that we find, when we compare the solutions of the first order abstract Cauchy problem in discrete time (i.e. (1) in case $\alpha=1$) with those of the first order abstract Cauchy problem in continuous time, namely

$$u'(t)=Au(t),\quad u(0)=u_0,\quad t\geq 0,$$

is that whereas the last is well-posed if and only if A generates a strongly continuous semigroup of operators, the former is well-posed if and only if A generates a discrete C-semigroup, where $C:=(I-A)^{-1}$.

After that, we establish a subordination principle (see Theorem 3.1.5), in which we prove that (α, ν)-resolvent sequences and C-semigroups are related. We remember that the notion of subordination, in continuous time, was introduced by Prüss [71] for the theory of integral equations and then used by Bazhlekova [23] in the theory of abstract fractional evolution equations.

This result extends and improves (2) and [6, Theorem 2.3]. Further, in order to show how our results apply in other contexts, we introduce a Hilfer fractional difference operator $\Delta^{\alpha,\beta}$ of order $\alpha > 0$ and type $0 \leq \beta \leq 1$. Then, we show the connection between (α, ν)-resolvent sequence and the solutions of the abstract Cauchy problem in discrete-time that involves the Hilfer fractional difference operator precisely mentioned.

On the other hand, the third problem is solved in the following way. We define and investigate a new class of vector-valued functions defined on $\mathbb{Z}$ called (N, λ)-periodic discrete functions (see Chapter 4). This type of sequences is the discrete version of the vector-valued (ω, c)-periodic functions introduced in [18]. Thus, we say that a function f is (N, λ)-periodic discrete function if there exist $N \in \mathbb{Z}_+$ and $\lambda \in \mathbb{C} \setminus \{0\}$ such that $f(n+N) = \lambda f(n)$ for all n integer. This definition includes: discrete periodic functions ($\lambda = 1$), discrete anti-periodic functions ($\lambda = -1$), discrete Bloch-periodic functions ($\lambda = e^{ikN}$) and unbounded functions ($|\lambda| \neq 1$).

Finally, we consider the problem of the existence and uniqueness of (N, λ)-periodic discrete solutions for (4). We have success into solve this problem by using fixed point techniques. See Section 4.2 below.

Finally, regarding the fourth problem, our main result regarding the solution of (5) says the following: Suppose that $1 \in \rho(A)$ and

$$r_A := \|(I - A)^{-1}\| < 1.$$

Assume that there exists $(N, \lambda) \in \mathbb{N} \times (\mathbb{C} \setminus \mathbb{D}(0, 1))$ and a constant $L > 0$ such that $f(n+N,\lambda x) = \lambda f(n,x)$ for all $(n,x) \in \mathbb{Z} \times X$ and

$$\|f(n,x) - f(n,y)\|_X \ \leq L\|x-y\|_X,$$

for all $x, y \in X$ and all $n \in \mathbb{Z}$. If

$$L < \left(1 - \frac{1}{|\lambda|^{1/N}}\right)^{\alpha} + \left(\frac{1}{r_A} - 1\right),$$

then equation (5) has a unique (N,λ)-periodic solution in a mild sense.

As methods, we use the technique of *resolvent sequences of operators,* mentioned. Thus, an explicit representation of the solutions of (5) can be obtained, which allows the use of fixed point theorem.

The thesis is organized in five chapters. The first chapter is devoted to the preliminaries, providing some basic definitions on continuous and discrete fractional calculus, and notation. Furthermore, we show a discrete version of the Mittag-Leffler sequences and demonstrate an interesting result that relates this definition to its continuous counterpart. The second chapter presents and investigates the properties of the discrete stable distribution α-Lévy and the discrete scaled Wright function. In addition, we focus on answer to our first open problem. The third chapter is dedicated to a new definition and main properties of the Hilfer operator, and its relationship with the operator already defined in the literature. Finally, Chapters 4 and 5 are dedicated to the detailed study of the third and fourth open problems described above, respectively.

1. Preliminaries

In this chapter, we present a summary of definitions and main results of the literature that will be used throughout this thesis.

1.1 Continuous fractional calculus

In this section, we recall some concepts and basic results about fractional calculus in continuous time.

We denote by Γ the Gamma function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt, \quad \operatorname{Re} x > 0.$$

It is well-known that the Gamma function can be extended to $\mathbb{C} \setminus \{0, -1, -2, ...\}$ by the following equality

$$\Gamma(x+1) = x\Gamma(x),$$

see [9]. Moreover, we recall the following asymptotic behavior of the Gamma function. Let $\alpha, z \in \mathbb{C}$, then

$$\frac{\Gamma(z+\alpha)}{\Gamma(z)} = z^\alpha \left(1 + \frac{\alpha(\alpha+1)}{2z} + O(|z|^{-2})\right), \quad |z| \to \infty, \tag{1.1}$$

whenever $z \neq 0, -1, -2, \ldots$, and $z \neq -\alpha, -\alpha - 1, \ldots$, see [74]. As a consequence of (1.1), observe that

$$\lim_{z\to\infty} \frac{\Gamma(z+\alpha)}{\Gamma(z) z^\alpha} = 1. \tag{1.2}$$

Let $\alpha > 0$ be given. We denote

$$g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t > 0.$$

Let $0 < \alpha < 1$ and f be a locally integrable function. The Riemann-Liouville fractional derivative of f of order α is given by

$$_{RL}D_t^{\alpha} f(t) := \frac{d}{dt}\int_0^t g_{1-\alpha}(t-s)f(s)\,ds,\ t \geq 0.$$

The Caputo fractional derivative of order α of a function f is defined by

$$_{C}D_t^{\alpha} f(t) := \int_0^t g_{1-\alpha}(t-s)f'(s)\,ds,\ t \geq 0,$$

where f' is the first order distributional derivative of $f(\cdot)$, for example if we assume that $f(\cdot)$ has locally integrable distributional derivative up to order one. Then, when $\alpha = 1$, we obtain $_{C}D_t^{\alpha} := \frac{d}{dt}$. For more details, see for example [63, 66, 69].

R. Hilfer in [46] introduced the concept of generalized Riemann–Liouville fractional derivative. This derivative of arbitrary order contains Riemann–Liouville and Caputo fractional derivatives as particular cases.

More precisely, the Hilfer fractional derivative of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$ for an absolutely integrable function u is defined by

$$_{H}D_t^{\alpha,\beta} u(t) := I_t^{\beta(1-\alpha)}\,{}_{RL}D_t^{\nu} u(t), \tag{1.3}$$

where $\nu := \alpha + \beta(1-\alpha)$ and $I_t^{\gamma} u(t) := \int_0^t g_{\gamma}(t-s)u(s)ds$.

Now, we recall the definitions and some properties of four special functions which play a significant role in the study of continuous fractional calculus.

The Mittag-Leffler functions are defined by

$$E_{\alpha,\beta}(s) := \sum_{n=0}^{\infty} \frac{s^n}{\Gamma(\alpha n + \beta)}, \quad \alpha,\ \beta > 0,\ s \in \mathbb{C}. \tag{1.4}$$

See [69]. We write $E_{\alpha}(s) := E_{\alpha,1}(s)$. They are solutions of the fractional differential problems

$$_{C}D_t^{\alpha} E_{\alpha}(\omega t^{\alpha}) = \omega E_{\alpha}(\omega t^{\alpha}),$$

and

$$ {}_{RL}D_t^{\alpha}\Big(t^{\alpha-1}E_{\alpha,\alpha}(\omega t^{\alpha})\Big) = \omega t^{\alpha-1}E_{\alpha,\alpha}(\omega t^{\alpha}), $$

for $0 < \alpha < 1$, under certain initial conditions.

For $\omega > 0$, the following property hold

$$ \int_0^{\infty} e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^{\alpha})\, dt = \frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha}-\omega}, \quad \operatorname{Re}\lambda > \omega^{\frac{1}{\alpha}}. \tag{1.5} $$

In the case that $\beta = 1$, (1.5) is the Laplace transform of Mittag-Leffler functions.

On the other hand, for $0 < \alpha < 2$, $\beta \in \mathbb{R}$, μ such that $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$ and C a real constant, the following estimate holds

$$ |E_{\alpha,\beta}(s)| \le \frac{C}{1+|s|}, \quad \mu \le |\arg(s)| \le \pi, \quad |s| \ge 0. \tag{1.6} $$

See [69, Theorem 1.6]. F. Mainardi [64] conjectured, and then Simon [72] proved that the following inequality holds:

$$ \frac{1}{1+\Gamma(1-\alpha)s} \le E_{\alpha}(-s) \le \frac{1}{1+\Gamma(1-\alpha)^{-1}s}, \quad 0 < \alpha < 1, \quad s > 0. $$

For more details about the Mittag-Leffler function $E_{\alpha,\beta}$, see [37, Chapter 18]. Another important function is the Lévy α-stable distribution (also called Lévy probability density function or stable Lévy process) which is defined as follows

$$ f_{t,\alpha}(\lambda) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{z\lambda - tz^{\alpha}}\, dz, \quad \sigma > 0,\ t > 0,\ \lambda \ge 0,\ 0 < \alpha < 1, \tag{1.7} $$

where the branch of z^{α} is taken such that $\operatorname{Re} z^{\alpha} > 0$ for $\operatorname{Re} z > 0$. This branch is single-valued in the z-plane and cut along the negative real axis, see [78, p.260-262].

Recall the definition of the Wright type function (see [63])

$$ W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\lambda n+\mu)} = \frac{1}{2\pi i}\int_{H_a} \sigma^{-\mu} e^{\sigma + z\sigma^{-\lambda}} d\sigma,\ \lambda > -1,\ \mu \ge 0,\ z \in \mathbb{C}, $$

where H_a denotes the Hankel path defined as a contour which starts and ends at $-\infty$ and encircles the origin once counterclockwise.

For $0 < \alpha < 1$ and $\beta \geq 0$, the scaled Wright function in two variables $\psi_{\alpha,\beta}$ (introduced by Abadias and Miana in [7]) is given by

$$\psi_{\alpha,\beta}(t,s) := t^{\beta-1} W_{-\alpha,\beta}(-st^{-\alpha}), \quad t>0, \ s\in\mathbb{C}. \tag{1.8}$$

An interesting fact is the connection between the Wright and the Mittag-Leffer functions:

$$\int_0^\infty e^{\lambda s}\psi_{\alpha,\beta}(t,s)\,ds = t^{\alpha+\beta-1}E_{\alpha,\beta}(\lambda t^\alpha), \quad t>0, \ \lambda\in\mathbb{C}. \tag{1.9}$$

On the other hand, the following identity holds:

$$\int_0^\infty g_\eta(s)\psi_{\alpha,\beta}(t,s)\,ds = g_{\alpha\eta+\beta}(t), \ t,\eta>0. \tag{1.10}$$

Many properties about such functions that we will use along the paper appear in [7].

Finally, we recall the definition of Fox H-functions (see [54]). Let $m,n,p,q\in\mathbb{N}_0$ such that $0\leq m\leq q$, $0\leq n\leq p$. Let $a_i, b_j\in\mathbb{C}$ and $\alpha_i,\beta_j\in\mathbb{R}_+ := (0,\infty)$. The Fox H-function $H^{m,n}_{p,q}$ is defined via a Mellin-Barnes type integral

$$H^{m,n}_{p,q}(z) := H^{m,n}_{p,q}\left[z \;\middle|\; \begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix} \right] = \frac{1}{2\pi i}\int_\gamma \mathcal{H}^{m,n}_{p,q}(s) z^{-s}\,ds,$$

where

$$(a_i,\alpha_i)_{1,p} := (a_1,\alpha_1),\cdots,(a_p,\alpha_p),$$
$$(b_j,\beta_j)_{1,q} := (b_1,\beta_1),\cdots,(b_q,\beta_q),$$

$$\mathcal{H}^{m,n}_{p,q}(s) = \frac{\displaystyle\prod_{j=1}^m \Gamma(b_j+\beta_j s)\prod_{i=1}^n \Gamma(1-a_i-\alpha_i s)}{\displaystyle\prod_{i=n+1}^p \Gamma(a_i+\alpha_i s)\prod_{j=m+1}^q \Gamma(1-b_j-\beta_j s)}$$

and γ is the infinite contour in the complex plane which separates the poles

$$b_{jl} = \frac{-b_j - l}{\beta_j} \; (j = 1, \cdots, m; \; l \in \mathbb{N}_0)$$

of the Gamma function $\Gamma(b_j + \beta_j s)$ to the left of γ and the poles

$$a_{ik} = \frac{1 - a_i + k}{\alpha_i} \; (i = 1, \cdots, n; \; k \in \mathbb{N}_0)$$

to the right of γ. The following identities hold:

$$H^{m,n}_{p,q}\left[\, z \;\middle|\; \begin{matrix} (a_i, \alpha_i)_{1,p-1}, (b_1, \beta_1) \\ (b_j, \beta_j)_{1,q} \end{matrix} \,\right] = H^{m-1,n}_{p-1,q-1}\left[\, z \;\middle|\; \begin{matrix} (a_i, \alpha_i)_{1,p-1} \\ (b_j, \beta_j)_{2,q} \end{matrix} \,\right] \tag{1.11}$$

and

$$H^{m,n}_{p,q}\left[\, z^{-1} \;\middle|\; \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \,\right] = H^{n,m}_{q,p}\left[\, z \;\middle|\; \begin{matrix} (1 - b_j, \beta_j)_{1,q} \\ (1 - a_i, \alpha_i)_{1,p} \end{matrix} \,\right]. \tag{1.12}$$

See Proposition 2.2 and Proposition 2.3 of [54].

1.2 Discrete fractional calculus

In this section, we recall the definition of Cesàro numbers and some useful properties of them. Further, we establish the definitions of the difference fractional operators which we will work.

Let X be a complex Banach space equipped with the norm $\|\cdot\|_X$ and $\mathcal{B}(X)$ denotes the Banach space of all bounded operators defined on X. For a real number a, we denote $\mathbb{N}_a := \{a, a+1, a+2, ...\}$ and when $a = 1$, we write $\mathbb{N}$ instead of $\mathbb{N}_1$. The vector space of all vector valued sequences $f : D \to X$ will be denoted by $s(D; X)$, where D can be $\mathbb{N}_0$ or $\mathbb{Z}$. The $\mathcal{Z}$-transform of a vector-valued sequence $f \in s(\mathbb{N}_0; X)$ is defined by

$$\widetilde{f}(z) := \sum_{j=0}^{\infty} f(j) z^{-j},$$

where z is a complex number. Let γ be a circle centered at the origin of the z-plane that encloses all poles of $\widetilde{f}(z)z^{n-1}$. We recall that the inverse $\mathcal{Z}$-transform of $\widetilde{f}(z)$ is defined by

$$f(n) = \frac{1}{2\pi i}\int_\gamma \widetilde{f}(z)z^{n-1}\,dz. \tag{1.13}$$

For more details about $\mathcal{Z}$-transforms and its inverse, see [35, Chapter 6]. For an arbitrary $\alpha \in \mathbb{C}$, we denote by $k^\alpha(n)$ the Cesàro numbers which are the Fourier coefficients of the holomorphic function $(1-w)^{-\alpha}$ on the unitary disc, that is,

$$\frac{1}{(1-w)^\alpha} = \sum_{n=0}^{\infty} k^\alpha(n)w^n, \qquad |\omega| < 1. \tag{1.14}$$

It is known that an equivalent expression of the Cesàro numbers is given by

$$k^\alpha(n) := \begin{cases} \dfrac{\alpha(\alpha+1)\cdot\cdots\cdot(\alpha+n-1)}{n!}, & n \in \mathbb{N}, \\ 1, & n = 0. \end{cases} \tag{1.15}$$

See [56, 81]. Note that $k^0(n) := \delta_0(n)$ is the Kronecker delta.

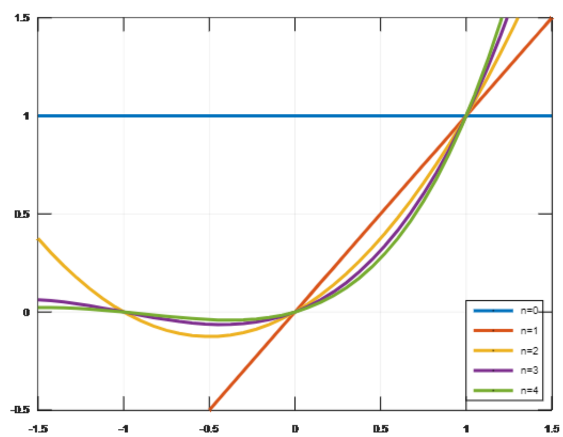

Figure 1.1: The graph of $\alpha \to k^\alpha(n)$ for $\alpha \in \mathbb{R}$ and $n = 0, 1, 2, 3, 4$.

Observe that, for $\alpha \in \mathbb{C} \setminus \{0, -1, -2, ...\}$, the sequence (1.15) is equivalent to

$$k^\alpha(n) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}, \tag{1.16}$$

and replacing $w = 1/z$, with $z \in \mathbb{C} \setminus \{0\}$, in (1.14), we find that the $\mathcal{Z}$-transform of k^α is given by the formula

$$\widetilde{k^\alpha}(z) = \left(\frac{z}{z-1}\right)^\alpha. \tag{1.17}$$

For $\alpha \in \{0\} \cup \mathbb{Z}^-$, the representation (1.17) is valid for all $z \in \mathbb{C} \setminus \{0\}$, while for all $\alpha \in \mathbb{C} \setminus \{0\} \cup \mathbb{Z}^-$ the expression is valid only for $|z| > 1$.

We recall the following properties of k^α, which appear for example in [40, 56, 81].

Proposition 1.2.1. *The following properties hold:*

(*i*) *For* $\alpha > 0$, $k^\alpha(n) > 0$, $n \in \mathbb{N}_0$.

(*ii*) *For all* $\alpha, \beta \in \mathbb{C}$ *and* $n \in \mathbb{N}_0$, *we have the semigroup property*

$$\sum_{j=0}^{n} k^\alpha(n-j)k^\beta(j) = k^{\alpha+\beta}(n). \tag{1.18}$$

(*iii*) *For* $\alpha > 0$,

$$k^\alpha(n) = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \quad n \in \mathbb{N}. \tag{1.19}$$

(*iv*) *For* $0 < \alpha < 1$,

$$k^\alpha(n+1) - k^\alpha(n) = k^{\alpha-1}(n+1), \quad n \in \mathbb{N}_0.$$

Now, we describe the discrete version of the Mittag-Leffler sequences and its properties.

Let $\alpha, \beta > 0$ and $\tau \in \mathbb{C}$. The Mittag-Leffler sequences (see [56, 67]) are given by

$$\mathcal{E}_{\alpha,\beta}(\tau, n) := \frac{1}{(n-1)!}\sum_{j=0}^{\infty} \frac{\Gamma(\alpha j + \beta + n - 1)}{\Gamma(\alpha j + \beta)}\tau^j, \quad n \in \mathbb{N}, \quad |\tau| < 1. \tag{1.20}$$

The convergence of previous series can be justified by (1.19). Using the Cesàro numbers (1.16), one can rewrite (1.20) as

$$\mathcal{E}_{\alpha,\beta}(\tau, n) = \sum_{j=0}^{\infty} k^{\alpha j+\beta}(n-1)\tau^j, \quad n \in \mathbb{N}, \quad |\tau| < 1.$$

Particularly, note that

$$\mathcal{E}_{1,1}(\tau,n)=\sum_{j=0}^{\infty}\tau^j k^{j+1}(n-1)=\sum_{j=0}^{\infty}\tau^j k^n(j)=(1-\tau)^{-n},\quad |\tau|<1.$$

The $\mathcal{Z}$-transform of the Mittag-Leffler sequence is given by

$$\sum_{n=0}^{\infty} z^{-n}\mathcal{E}_{\alpha,\beta}(\tau,n+1)=\left(\frac{z}{z-1}\right)^{\beta}\left(1-\tau\left(\frac{z}{z-1}\right)^{\alpha}\right)^{-1}, \tag{1.21}$$

where $|z|>1$. For more details, see e.g. [56, 67]. Now, we introduce the following function:

$$L_{\alpha,\beta}(\tau,t):=t^{\beta-1}E_{\alpha,\beta}(\tau t^{\alpha}),\quad t\geq 0,\quad \tau\in\mathbb{C},\quad \alpha,\beta>0,$$

and we show the following remarkable result that relates the discrete and continuous Mittag-Leffler functions by means of the Poisson transform.

Initially, we remember the definition of the Poisson transform. For each $n\in\mathbb{N}_0$, the Poisson distribution (with parameter t) is defined by

$$p_n(t):=e^{-t}\frac{t^n}{n!},\quad t\geq 0.$$

Given a continuous function $u:[0,\infty)\to X$ the Poisson transform of u was defined in [56] by

$$\mathcal{P}(u)(n):=\int_0^{\infty}p_n(t)u(t)dt,\ n\in\mathbb{N}_0. \tag{1.22}$$

Theorem 1.2.2. *For all $\tau\in\mathbb{C}$ we have*

$$\mathcal{P}(L_{\alpha,\beta}(\tau,\cdot))(n)=\mathcal{E}_{\alpha,\beta}(\tau,n+1),\quad n\in\mathbb{N}_0.$$

Proof. An easy calculation, using the definitions, shows the following identities

$$\mathcal{P}(L_{\alpha,\beta}(\tau,\cdot))(n)=\int_0^{\infty}e^{-t}\frac{t^n}{n!}\sum_{j=0}^{\infty}\frac{\tau^j t^{\alpha j+\beta-1}}{\Gamma(\alpha j+\beta)}dt=\sum_{j=0}^{\infty}\Big[\int_0^{\infty}e^{-t}\frac{t^{n+\alpha j+\beta-1}}{n!\Gamma(\alpha j+\beta)}\Big]\tau^j.$$

We assume that the exchanges between integrals and series are legitimate in view of the analyticity properties of the involved functions. Using the identity

$$\int_0^\infty e^{-wt}\frac{t^{\gamma-1}}{\Gamma(\gamma)}dt = \frac{1}{w^{\gamma-1}},$$

with $w = 1$ and $\gamma = n + \alpha j + \beta$, we obtain

$$\sum_{j=0}^{\infty}\Big[\int_0^\infty e^{-t}\frac{t^{n+\alpha j+\beta-1}}{n!\Gamma(\alpha j+\beta)}\Big]\tau^j = \sum_{j=0}^{\infty}\frac{\Gamma(\alpha j+n+\beta)}{n!\Gamma(\alpha j+\beta)}\tau^j = \sum_{j=0}^{\infty}\tau^j k^{\alpha j+\beta}(n).$$

$\square$

One of the key properties of the Poisson transform is the following relationship between the kernel function g_α and the kernel sequence k^α :

$$\mathcal{P}(g_\alpha)(n) = k^\alpha(n).$$

Additional properties of the Poisson transform are given in [6, Section 4].

In this work, we develop the discrete fractional calculus theory on specific discrete time scales: $\mathbb{N}_a$ and $\mathbb{Z}$.

Let u be a sequence defined on $\mathbb{N}_0$ (or $\mathbb{Z}$). The forward Euler operator Δ is defined by

$$(\Delta u)(n) := u(n+1) - u(n), \quad n \in \mathbb{N}_0 \text{ (or } \mathbb{Z}),$$

and the backward Euler operator ∇ of sequence u is defined by

$$(\nabla u)(n) := u(n) - u(n-1), \quad n \in \mathbb{N} \text{ (or } \mathbb{Z}).$$

Remark 1.2.3. *For convention, we use $\Delta u(n) := (\Delta u)(n)$ and $\nabla u(n) := (\nabla u)(n)$.*

Based on the works done by Atici and Eloe [20, 21], we consider the following definition of fractional backward sum (or sum of arbitrary order) and fractional backward difference operators (in the sense of Riemann-Liouville and Caputo).

Definition 1.2.4. *Let $f \in s(\mathbb{N}_0, X)$. For $\alpha \geq 0$, the α-th fractional backward sum of f is defined by means of the formula*

$$\nabla^{-\alpha} f(n) := \sum_{j=1}^{n} k^{\alpha}(n-j) f(j), \quad n \in \mathbb{N}.$$

Definition 1.2.5. *Let $0 < \alpha < 1$ and $f \in s(\mathbb{N}_0, X)$. The α-th Riemann-Liouville fractional backward difference of f is defined by*

$$_{RL}\nabla^{\alpha} f(n) := \nabla \nabla^{-(1-\alpha)} f(n), \quad n \in \mathbb{N},$$

and the Caputo fractional backward difference of order α is defined by

$$_{C}\nabla^{\alpha} f(n) := \nabla^{-(1-\alpha)} \nabla f(n), \quad n \in \mathbb{N}.$$

Now, for the fractional forward sum and fractional forward difference operators, we consider the definitions given by Lizama in [55, 56].

Definition 1.2.6. *Let $f \in s(\mathbb{N}_0, X)$. For $\alpha \geq 0$, the α-th fractional forward sum of f is defined by means of the formula*

$$\Delta^{-\alpha} f(n) := \sum_{j=0}^{n} k^{\alpha}(n-j) f(j), \quad n \in \mathbb{N}_0.$$

Definition 1.2.7. *Let $0 < \alpha < 1$ and $f \in s(\mathbb{N}_0, X)$. The α-th Riemann-Liouville fractional forward difference of f is defined by*

$$_{RL}\Delta^{\alpha} f(n) := \Delta \Delta^{-(1-\alpha)} f(n), \quad n \in \mathbb{N}_0,$$

and the Caputo fractional forward difference of order α is defined by

$$_{C}\Delta^{\alpha} f(n) := \Delta^{-(1-\alpha)} \Delta f(n), \quad n \in \mathbb{N}_0.$$

Finally, concerning the case of sequences defined on the set $\mathbb{Z}$ we recall fractional sum operator and fractional difference in the Weyl-like sense which was introduced by Abadias and Lizama [5], as follows.

Definition 1.2.8. *Let $f \in s(\mathbb{Z}, X)$. For $\alpha \geq 0$, the α-th fractional sum in the Weyl-like sense of f is defined by means of the formula*

$$\Delta_W^{-\alpha} f(n) := \sum_{j=-\infty}^{n} k^{\alpha}(n-j) f(j), \quad n \in \mathbb{Z}.$$

Definition 1.2.9. *Let $0 < \alpha < 1$ and $f \in s(\mathbb{Z}, X)$. The α-th fractional difference in the Weyl-like sense of f is defined by*

$$\Delta_W^{\alpha} f(n) := \Delta \Delta_W^{-(1-\alpha)} f(n), \quad n \in \mathbb{Z}.$$

2. Fundamental solutions and large-time behaviour for a discrete in time fractional diffusion equation

This chapter discusses asymptotic behaviour for the solutions of the fractional version of the discrete in time d-dimensional diffusion equation, which involves the Caputo fractional backward difference. For this purpose we introduce and investigate the properties the discrete Lévy α–stable distribution and the discrete scaled Wright function. Moreover, we prove that a solution of the fractional equation mentioned has an representation involving the discrete in time Gaussian kernel and the discrete scaled Wright function.

2.1 Some special functions in the discrete setting

In this section, we introduce a discrete version of the stable Lévy process and the scaled Wright functions. Further, we present some interesting properties which will be useful along the this work.

Definition 2.1.1. *Let $0 < \alpha \leq 1$ be given. For $n \in \mathbb{N}$, the discrete Lévy α-stable distribution is defined by*

$$l_\alpha(n,j) := \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_\Gamma z^{n-1}\left(\frac{z^\alpha-(z-1)^\alpha}{z^\alpha}\right)^j dz, & j \in \mathbb{N}, \\ \\ \delta_0(n), & j = 0, \end{cases} \tag{2.1}$$

where Γ *is a path oriented counterclockwise that encloses all the singularities of the complex variable function* $z \to z^{-\alpha j}(z^{\alpha} - (z-1)^{\alpha})^{j}$ *and* $\delta_0(n)$ *is the Kronecker delta.*

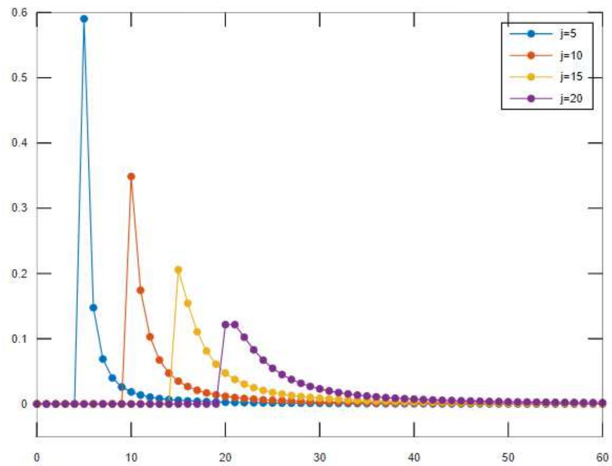

Figure 2.1: The discrete Lévy α-stable distribution for $\alpha = 0.9$ and $0 \leq n \leq 60$

Denote by $\mathbb{D}(a, r) \subset \mathbb{C}$ the open disk of center $a \in \mathbb{C}$ and radius $r > 0$. The following lemma will be very useful in what follows.

Lemma 2.1.2. *Let* $0 < \alpha \leq 1$ *be given and* $z \in \mathbb{D}(1, 1)$. *Then* $z^{\alpha} \in \mathbb{D}(1, 1)$.

Proof. Let $z = re^{i\theta}$. By hypothesis, $|\theta| < \pi/2$. We first claim that $(\cos\theta)^{\alpha} < \cos(\alpha\theta)$. In fact, it is enough to prove the claim for $0 < \theta < \pi/2$ (since $\cos x$ is even). Since $\cos x$ is positive and decreasing on the interval $[0, \pi/2]$ and $0 < \alpha\theta < \theta$, we have $\cos(\alpha\theta) > \cos\theta > 0$. Therefore $\ln(\cos\theta) < \ln(\cos\alpha\theta)$. Hence, the condition $0 < \alpha < 1$ implies $\alpha \ln(\cos\theta) < \ln(\cos\alpha\theta)$. This proves the claim. Now, using the claim and the inequality $2^{\alpha} \leq 2$, we obtain $2^{\alpha}(\cos\theta)^{\alpha} \leq 2\cos(\alpha\theta)$. It shows that if $r < 2\cos\theta$ then $r^{\alpha} < 2\cos(\alpha\theta)$, or, equivalently, that $|1 - z| < 1$ implies $|1 - z^{\alpha}| < 1$, proving the lemma. $\square$

Remark 2.1.3. *We observe that the integral on the right hand side of* (2.1) *contains the analytic function* $z \to \dfrac{z^{\alpha} - (z-1)^{\alpha}}{z^{\alpha}} = 1 - \left(1 - \dfrac{1}{z}\right)^{\alpha}$. *Suppose*

$|z| > 1$. *Then* $1-\dfrac{1}{z} \in \mathbb{D}(1,1)$. *By the above lemma, we obtain* $\left(1-\dfrac{1}{z}\right)^{\alpha} \in \mathbb{D}(1,1)$. *We conclude that*

$$\left|1-\left(1-\frac{1}{z}\right)^{\alpha}\right| < 1 \quad \text{for all } |z| > 1. \tag{2.2}$$

Next, we present some fundamental properties of the discrete Lévy α-stable distribution.

Proposition 2.1.4. *Let* $0 < \alpha \leq 1$, $j \in \mathbb{N}$ *and* $n \in \mathbb{N}_0$ *be given. The following properties hold:*

(i) $\widetilde{l_{\alpha}}(z,j) = (1-\widetilde{k^{-\alpha}}(z))^{j}$, *for all* $|z| > 1$.

(ii) $l_{\alpha}(n,j) = \displaystyle\sum_{i=0}^{j} \binom{j}{i} (-1)^{i} k^{-\alpha i}(n)$.

(iii) $0 \leq l_{\alpha}(n,j)$.

(iv) $\displaystyle\sum_{i=0}^{\infty} l_{\alpha}(i,j) = 1$.

(v) $l_{\alpha}(n,j+2) - 2l_{\alpha}(n,j+1) + l_{\alpha}(n,j) = (k^{-2\alpha} * l_{\alpha}(\cdot,j))(n)$.

(vi) *Let* $\omega > 1$ *be given. Then*

$$\sum_{i=0}^{\infty} \widetilde{l_{\alpha}}(z,i)\omega^{-i} = \omega \widehat{L}_{\alpha,\alpha}(1-\omega,\cdot)(1-1/z), \quad |z| > 1.$$

(vii) $\mathcal{P}(\widehat{f}_{\cdot,\alpha}(\zeta))(j-1) = \widetilde{l_{\alpha}}(z,j)$ *where* $|z| > 1$ *and* $\zeta := \left(\dfrac{(z-1)^{\alpha}}{z^{\alpha}-(z-1)^{\alpha}}\right)^{1/\alpha}$.

Proof.

(i) Note that, using (1.17) we obtain the identity

$$\begin{aligned} l_{\alpha}(n,j) &= \frac{1}{2\pi i}\int_{\Gamma} z^{n-1}\left(\frac{z^{\alpha}-(z-1)^{\alpha}}{z^{\alpha}}\right)^{j} dz \\ &= \frac{1}{2\pi i}\int_{\Gamma} z^{n-1}\left(1-\widetilde{k^{-\alpha}}(z)\right)^{j} dz. \end{aligned}$$

Then, by the inverse $\mathcal{Z}$-transform, we get the claimed property.

(*ii*) Applying the general binomial theorem (see [42, Formula 1.111]) on (*i*), we get

$$\widetilde{l}_\alpha(z,j) = \sum_{i=0}^{j} \binom{j}{i} (-1)^i \widetilde{k^{-\alpha i}}(z),$$

where we have used the group property (1.18) of the sequence kernel k^β to deduce that $\widetilde{k^{-\alpha i}}(z) = [\widetilde{k^{-\alpha}}(z)]^i$. Then the result follows by an application of the inverse $\mathcal{Z}$-transform.

(*iii*) By (*ii*), we have

$$l_\alpha(n,1) = k^0(n) - k^{-\alpha}(n) = \frac{\alpha(1-\alpha)(2-\alpha)\dots(n-1-\alpha)}{n!},$$

and, since $0 < \alpha \leq 1$, we deduce $l_\alpha(n,1) \geq 0$ for all $n \in \mathbb{N}$. Moreover, $l_\alpha(0,1) = 0$ by (1.15). On the other hand, observe that

$$1-2\alpha \leq 1-\alpha,\ 2-2\alpha \leq 2-\alpha,\ 3-2\alpha \leq 3-\alpha, \dots n-1-2\alpha \leq n-1-\alpha.$$

Since $i - \alpha > 0$ for all $i = 1, ..., n-1$, then

$$\prod_{i=1}^{n-1}(i-2\alpha) \leq \prod_{i=1}^{n-1}(i-\alpha). \tag{2.3}$$

Multiplying by $-2\alpha/n!$ in (2.3), we get

$$-\frac{2\alpha}{n!}\prod_{i=1}^{n-1}(i-2\alpha) \geq -\frac{2\alpha}{n!}\prod_{i=1}^{n-1}(i-\alpha)$$

or

$$\frac{(-2\alpha)(1-2\alpha)\dots(n-1-2\alpha)}{n!} \geq 2\frac{(-\alpha)(1-\alpha)\dots(n-1-\alpha)}{n!}.$$

Using this last inequality together with the definition of k^α, we have

$$k^{-2\alpha}(n) \geq 2k^{-\alpha}(n).$$

Now, note from (*ii*) that

$$l_\alpha(n,2) = k^0(n) - 2k^{-\alpha}(n) + k^{-2\alpha}(n).$$

From which we deduce

$$l_\alpha(n,2) = -2k^{-\alpha}(n) + k^{-2\alpha}(n) \geq -2k^{-\alpha}(n) + 2k^{-\alpha}(n) = 0, \quad n \in \mathbb{N},$$

and $l_\alpha(0,2) = 0$ by (1.15). Now, observe that from the previous calculations and (the proof of) (i), we obtain

$$\begin{aligned} l_\alpha(n,3) &= \frac{1}{2\pi i}\int_\Gamma z^{n-1}(1-\widetilde{k^{-\alpha}}(z))^3\, dz \\ &= \frac{1}{2\pi i}\int_\Gamma z^{n-1}(1-\widetilde{k^{-\alpha}}(z))^2(1-\widetilde{k^{-\alpha}}(z))\, dz \\ &= \frac{1}{2\pi i}\int_\Gamma z^{n-1}\widetilde{l_\alpha}(n,2)\widetilde{l_\alpha}(n,1)\, dz \\ &= \sum_{p=0}^{n} l_\alpha(n-p,2)l_\alpha(p,1) \geq 0, \quad n \in \mathbb{N}_0. \end{aligned}$$

Now, assume that for $j = m \in \mathbb{N}$ we have

$$l_\alpha(n,m) \geq 0, \qquad n \in \mathbb{N}_0.$$

Then, for $j = m+1$ and proceeding as in the case $j = 3$, we obtain

$$l_\alpha(n,m+1) = \sum_{p=0}^{n} l_\alpha(n-p,m)l_\alpha(p,1) \geq 0,$$

for all $n \in \mathbb{N}$. This proves the claim.

(iv) Let $z \in \mathbb{R}$ be given. Note that the claimed identity is a particular case of (i) by letting $z \to 1^+$ and taking into account that $\widetilde{k^{-\alpha}}(1) = 0$, again by (1.17).

(v) Note that,

$$\begin{aligned} &l_\alpha(n,j+2) - 2l_\alpha(n,j+1) + l_\alpha(n,j) \\ &\quad = \frac{1}{2\pi i}\int_\Gamma z^{n-1}\left[(1-\widetilde{k^{-\alpha}}(z))^{j+2} - 2(1-\widetilde{k^{-\alpha}}(z))^{j+1} + (1-\widetilde{k^{-\alpha}}(z))^{j}\right]\, dz \\ &\quad = \frac{1}{2\pi i}\int_\Gamma z^{n-1}\widetilde{k^{-2\alpha}}(z)\widetilde{l_\alpha}(z,j)\, dz. \end{aligned}$$

The property follows by applying the inverse $\mathcal{Z}$-transform for the finite convolution.

(*vi*) Using geometric series and Remark 2.1.3 we find, for all $\omega > 1$, that

$$\sum_{i=0}^{\infty} \widetilde{l}_\alpha(z,i)\omega^{-i} = \sum_{i=0}^{\infty} \left(\frac{z^\alpha - (z-1)^\alpha}{z^\alpha}\right)^i \omega^{-i} = \frac{\omega z^\alpha}{(z-1)^\alpha + z^\alpha(\omega-1)}.$$

On the other hand, by the Laplace transform property (1.5), we have

$$\omega \widehat{L}_{\alpha,\alpha}(-(\omega-1),\cdot)(\lambda) = \frac{\omega}{\lambda^\alpha + (\omega-1)}.$$

Evaluating at $\lambda = 1 - 1/z$, we obtain

$$\omega \widehat{L}_{\alpha,\alpha}(-(\omega-1),\cdot)(1-1/z) = \frac{\omega}{(1-1/z)^\alpha + (\omega-1)} = \frac{\omega}{\left(\frac{z-1}{z}\right)^\alpha + (\omega-1)}$$
$$= \frac{\omega z^\alpha}{(z-1)^\alpha + z^\alpha(\omega-1)}.$$

(*vii*) By (1.7), we have $\widehat{f}_{s,\alpha}(\zeta) = e^{-s\zeta^\alpha}$ for all $\operatorname{Re}\zeta > 0$. Then, using the Poisson transform, we get

$$\mathcal{P}(\widehat{f}_{\cdot,\alpha}(\zeta))(j-1) = \int_0^\infty \frac{s^{j-1}}{(j-1)!} e^{-(1+\zeta^\alpha)s}\, ds = \frac{1}{(1+\zeta^\alpha)^j}.$$

Define $\zeta := \dfrac{z-1}{(z^\alpha - (z-1)^\alpha)^{1/\alpha}}$ where $|z| > 1$. Hence

$$\mathcal{P}(\widehat{f}_{\cdot,\alpha}(\zeta))(j-1) = \left(\frac{1}{1 + \dfrac{(z-1)^\alpha}{z^\alpha - (z-1)^\alpha}}\right)^j = \left(\frac{z^\alpha - (z-1)^\alpha}{z^\alpha}\right)^j = \widetilde{l}_\alpha(z,j).$$

□

Next, let us define the discrete scaled Wright function.

Definition 2.1.5. *Let $0 < \alpha < 1$ and $0 \le \beta$ be given. For $n \in \mathbb{N}_0$ and $h > 0$, the discrete scaled Wright function $\varphi^h_{\alpha,\beta}$ is defined by*

$$\varphi^h_{\alpha,\beta}(n,j) := \frac{1}{2\pi i}\int_\Upsilon \frac{1}{z^{n+1}} \frac{\left(1 - h(\frac{1-z}{h})^\alpha\right)^j}{(\frac{1-z}{h})^\beta}\, dz, \quad j \in \mathbb{N}_0, \tag{2.4}$$

where Υ is the path oriented counterclockwise given by the circle centered at the origin and radius $0 < r < 1$. When $h = 1$ we write $\varphi^h_{\alpha,\beta}(n,j) \equiv \varphi_{\alpha,\beta}(n,j)$.

Remark 2.1.6. *Note that if* $|z| < 1$*, then* $\frac{1-z}{h}$ *belongs to the disc centered at* $1/h$ *and radius* $1/h$*. So, for each* $j \in \mathbb{N}_0$*, the function* $z \to \frac{\left(1-h(\frac{1-z}{h})^{\alpha}\right)^j}{(\frac{1-z}{h})^{\beta}}$ *is holomorphic on the unit disc. Therefore, by the Cauchy formula for the derivatives, we have defined* $\varphi^h_{\alpha,\beta}(n,j)$ *as the* n*-coefficient of the power series centered at the origin of such holomorphic function.*

In the following proposition, we present some useful properties of the discrete scaled Wright function $\varphi^h_{\alpha,\beta}$. Many of them follow the spirit of the analogue ones in the continuous case, see [7, Theorem 3].

Proposition 2.1.7. *Let* $0 < \alpha < 1$*,* $0 \leq \beta$*,* $0 < h$ *and* $n, j \in \mathbb{N}_0$*. The following properties hold:*

(i) $\varphi^h_{\alpha,\beta}(n,j) = h^{\beta} \sum_{i=0}^{j} \binom{j}{i} (-1)^i h^{i-\alpha i} k^{\beta-\alpha i}(n).$

(ii) $\varphi^h_{\alpha,\beta+\gamma}(n,j) = h^{\beta} \sum_{i=0}^{n} k^{\beta}(n-i) \varphi^h_{\alpha,\gamma}(i,j), \quad \gamma > 0.$

(iii) *For* $t > 0$*,*
$$\frac{1}{h^n \Gamma(n)} \int_0^{\infty} e^{-s/h} s^{n-1} \psi_{\alpha,\beta}(s,t)\, ds = e^{-t/h} \sum_{j=1}^{\infty} \varphi^h_{\alpha,\beta}(n-1, j-1) \frac{t^{j-1}}{h^j (j-1)!},$$
where $\psi_{\alpha,\beta}$ *is given by* (1.8).

(iv) $\sum_{j=1}^{\infty} \varphi_{\alpha,\beta}(n-1, j-1)\, (1-\tau)^{-j} = \mathcal{E}_{\alpha,\alpha+\beta}(\tau, n),$
$n \in \mathbb{N},\ |\tau| < 1.$

(v) $\varphi^h_{\alpha,\beta}(n,j) - \varphi^h_{\alpha,\beta}(n,j+1) = h\varphi^h_{\alpha,\beta-\alpha}(n,j).$

(vi) $\varphi^h_{\alpha,0}(n,j+1) = \sum_{p=0}^{n} \varphi^h_{\alpha,0}(n-p,j) \varphi^h_{\alpha,0}(p,1).$

(vii) $\varphi^h_{\alpha,\beta}(n,j) \geq 0, \quad 0 < h \leq 1.$

(viii) $\sum_{i=0}^{\infty} \varphi^h_{\alpha,0}(i,j) = 1.$

(ix) $\displaystyle\sum_{j=0}^{\infty} \varphi^h_{\alpha,\beta}(n,j)k^{\gamma}(j) = h^{\beta+\gamma(\alpha-1)}k^{\beta+\gamma\alpha}(n).$

Proof.

(i) Note that for $|z|<1$, we can write

$$h^{\beta}\sum_{i=0}^{j}\binom{j}{i}(-1)^i h^{i-\alpha i}(1-z)^{\alpha i-\beta} = \frac{1}{(\frac{1-z}{h})^{\beta}}\sum_{i=0}^{j}\binom{j}{i}(-1)^i h^i\left(\frac{1-z}{h}\right)^{\alpha i} = \frac{\left(1-h(\frac{1-z}{h})^{\alpha}\right)^j}{(\frac{1-z}{h})^{\beta}}.$$

By the uniqueness of the coefficients, we have the result.

(ii) The identity follows from the previous item (i) and (1.18). Indeed,

$$\begin{aligned}\varphi^h_{\alpha,\beta+\gamma}(n,j) =& h^{\beta+\gamma}\sum_{w=0}^{j}\binom{j}{w}(-1)^i h^{w-\alpha w}k^{\beta+\gamma-\alpha w}(n)\\ =& h^{\beta+\gamma}\sum_{w=0}^{j}\binom{j}{w}(-1)^i h^{w-\alpha w}\sum_{i=0}^{n}k^{\beta}(n-i)k^{\gamma-\alpha w}(i)\\ =& h^{\beta+\gamma}\sum_{i=0}^{n}k^{\beta}(n-i)\sum_{w=0}^{j}\binom{j}{w}(-1)^w h^{w-\alpha w}k^{\gamma-\alpha w}(i)\\ =& h^{\beta}\sum_{i=0}^{n}k^{\beta}(n-i)\varphi^h_{\alpha,\gamma}(i,j).\end{aligned}$$

(iii) Note that, by (1.8)

$$\begin{aligned}&\frac{1}{h^n\Gamma(n)}\int_0^{\infty} e^{-s/h}s^{n-1}\psi_{\alpha,\beta}(s,t)\,ds\\ &\qquad= \frac{1}{h^n\Gamma(n)}\int_0^{\infty} e^{-s/h}s^{n+\beta-2}W_{-\alpha,\beta}(-ts^{-\alpha})\,ds\\ &\qquad= \frac{1}{h^n}\sum_{i=0}^{\infty}\frac{(-t)^i}{\Gamma(n)\Gamma(-\alpha i+\beta)i!}\int_0^{\infty} e^{-s/h}s^{n+\beta-2-\alpha i}\,ds\\ &\qquad= h^{\beta-1}\sum_{i=0}^{\infty}h^{-\alpha i}\frac{\Gamma(n-1+\beta-\alpha i)}{\Gamma(n)\Gamma(-\alpha i+\beta)}\frac{(-t)^i}{i!}\\ &\qquad= h^{\beta-1}\sum_{i=0}^{\infty}h^{-\alpha i}k^{\beta-\alpha i}(n-1)\frac{(-t)^i}{i!}.\end{aligned}$$

On the other hand, by (i) we obtain

$$
\begin{aligned}
\sum_{j=0}^{\infty} \varphi^h_{\alpha,\beta}(n-1,j)\frac{t^j}{h^{j+1}j!} &= h^{\beta-1}\sum_{j=0}^{\infty}\sum_{i=0}^{j}\binom{j}{i}(-1)^i h^{i-\alpha i}k^{\beta-\alpha i}(n-1)\left(\frac{t}{h}\right)^j\frac{1}{j!} \\
&= h^{\beta-1}\sum_{i=0}^{\infty}\sum_{j=i}^{\infty}\binom{j}{i}(-1)^i h^{i-\alpha i}k^{\beta-\alpha i}(n-1)\left(\frac{t}{h}\right)^j\frac{1}{j!} \\
&= h^{\beta-1}\sum_{i=0}^{\infty}(-1)^i h^{i-\alpha i}k^{\beta-\alpha i}(n-1)\sum_{j=i}^{\infty}\binom{j}{i}\left(\frac{t}{h}\right)^j\frac{1}{j!} \\
&= h^{\beta-1}\sum_{i=0}^{\infty}(-1)^i h^{i-\alpha i}k^{\beta-\alpha i}(n-1)\sum_{j=i}^{\infty}\frac{1}{i!(j-i)!}\left(\frac{t}{h}\right)^j \\
&= h^{\beta-1}\sum_{i=0}^{\infty}(-1)^i h^{i-\alpha i}k^{\beta-\alpha i}(n-1)\sum_{j=0}^{\infty}\frac{1}{i!j!}\left(\frac{t}{h}\right)^{j+i} \\
&= h^{\beta-1}\sum_{i=0}^{\infty}(-1)^i h^{-\alpha i}k^{\beta-\alpha i}(n-1)\frac{t^i}{i!}e^{t/h},
\end{aligned}
$$

for all $n \in \mathbb{N}$. Thus, the result is proved.

(iv) Let $\tau \in \mathbb{C}$ such that $|\tau| < 1$. Then,

$$
\begin{aligned}
&\sum_{j=1}^{\infty}\varphi_{\alpha,\beta}(n-1,j-1)\,(1-\tau)^{-j} \\
&\qquad = \sum_{j=1}^{\infty}\varphi_{\alpha,\beta}(n-1,j-1)\int_0^{\infty} e^{-(1-\tau)s}\frac{s^{j-1}}{(j-1)!}ds \\
&\qquad = \int_0^{\infty} e^{-(1-\tau)s}\sum_{j=1}^{\infty}\varphi_{\alpha,\beta}(n-1,j-1)\frac{s^{j-1}}{(j-1)!}ds \\
&\qquad = \frac{1}{\Gamma(n)}\int_0^{\infty} e^{\tau s}\int_0^{\infty} e^{-t}t^{n-1}\psi_{\alpha,\beta}(t,s)\;dt\;ds \\
&\qquad = \frac{1}{\Gamma(n)}\int_0^{\infty} e^{-t}t^{n-1}t^{\alpha+\beta-1}E_{\alpha,\alpha+\beta}(\tau t^{\alpha})\;dt \\
&\qquad = \mathcal{E}_{\alpha,\alpha+\beta}(\tau,n),
\end{aligned}
$$

where we have used item (iii), (1.9) and Theorem 1.2.2.

(v) We recall that

$$
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \tag{2.5}
$$

See [14, Section 1.4]. Then, the result is obtained as follows:

$$\begin{aligned}
\varphi^h_{\alpha,\beta}(n,j)-&\varphi^h_{\alpha,\beta}(n,j+1)\\
&= h^{\beta}\sum_{i=0}^{j}\binom{j}{i}(-1)^i h^{i-\alpha i}k^{\beta-\alpha i}(n)\\
&\qquad - h^{\beta}\sum_{i=0}^{j+1}\binom{j+1}{i}(-1)^i h^{i-\alpha i}k^{\beta-\alpha i}(n)\\
&= -h^{\beta}\sum_{i=1}^{j+1}\binom{j}{i-1}(-1)^i h^{i-\alpha i}k^{\beta-\alpha i}(n)\\
&= h^{\beta+1-\alpha}\sum_{i=0}^{j}\binom{j}{i}(-1)^i h^{i-\alpha i}k^{\beta-\alpha(i+1)}(n)\\
&= h\varphi^h_{\alpha,\beta-\alpha}(n,j).
\end{aligned}$$

(vi) By item (i), (2.5) and (1.18) it follows

$$\begin{aligned}
\varphi^h_{\alpha,0}(n,j+1) &= \sum_{i=0}^{j+1}\binom{j+1}{i}(-1)^i h^{i-\alpha i}k^{-\alpha i}(n)\\
&= \sum_{i=0}^{j}\binom{j}{i}(-1)^i h^{i-\alpha i}k^{-\alpha i}(n) + \sum_{i=1}^{j+1}\binom{j}{i-1}(-1)^i h^{i-\alpha i}k^{-\alpha i}(n)\\
&= \sum_{i=0}^{j}\binom{j}{i}(-1)^i h^{i-\alpha i}k^{-\alpha i}(n) - \sum_{i=0}^{j}\binom{j}{i}(-1)^i h^{i-\alpha i}h^{1-\alpha}k^{-\alpha(i+1)}(n)\\
&= \sum_{p=0}^{n}\sum_{i=0}^{j}\binom{j}{i}(-1)^i h^{i-\alpha i}k^{-\alpha i}(n-p)k^0(p)\\
&\qquad - \sum_{p=0}^{n}\sum_{i=0}^{j}\binom{j}{i}(-1)^i h^{i-\alpha i}h^{1-\alpha}k^{-\alpha i}(n-p)k^{-\alpha}(p)\\
&= \sum_{p=0}^{n}\sum_{i=0}^{j}\binom{j}{i}(-1)^i h^{i-\alpha i}k^{-\alpha i}(n-p)\left(k^0(p)-h^{1-\alpha}k^{-\alpha}(p)\right)\\
&= \sum_{p=0}^{n}\sum_{i=0}^{j}\binom{j}{i}(-1)^i h^{i-\alpha i}k^{-\alpha i}(n-p)\sum_{i=0}^{1}\binom{1}{i}(-1)^i h^{i-\alpha i}k^{-\alpha i}(p)\\
&= \sum_{p=0}^{n}\varphi^h_{\alpha,0}(n-p,j)\varphi^h_{\alpha,0}(p,1).
\end{aligned}$$

(vii) From Definition 2.1.5, we have that $\varphi^h_{\alpha,0}(n,0)=\delta_0(n)$ for $n\in\mathbb{N}_0$. Furthermore,

$$\varphi^h_{\alpha,0}(n,1)=k^0(n)-h^{1-\alpha}k^{-\alpha}(n).$$

Then,

$$\varphi^h_{\alpha,0}(0,1)=1-h^{1-\alpha}\geq 0$$

and

$$\varphi^h_{\alpha,0}(n,1)=h^{1-\alpha}\frac{\alpha(1-\alpha)(2-\alpha)\cdots(n-1-\alpha)}{n!}\geq 0,\ \ n\in\mathbb{N}.$$

By (i) of the Proposition 1.2.1 and items (vi) and (ii) the result follows.

$(viii)$ The identity is a particular case of (2.4), by letting $z\to 1^-$ with $z\in\mathbb{R}$.

(ix) By (2.4) and (1.17), we have

$$\begin{aligned}\sum_{l=0}^{\infty}\varphi^h_{\alpha,\beta}(n,l)k^{\gamma}(j)&=\frac{1}{2\pi i}\int_{\Upsilon}\frac{1}{z^{n+1}}\sum_{l=0}^{\infty}\frac{\left(1-h(\frac{1-z}{h})^{\alpha}\right)^j}{(\frac{1-z}{h})^{\beta}}k^{\gamma}(j)\,dz\\&=\frac{1}{h^{\gamma}}\frac{1}{2\pi i}\int_{\Upsilon}\frac{1}{z^{n+1}}\frac{1}{(\frac{1-z}{h})^{\beta+\gamma\alpha}}\,dz\\&=h^{\beta+\gamma(\alpha-1)}k^{\beta+\gamma\alpha}(n).\end{aligned}$$

□

Remark 2.1.8.

(i) *For $\beta=0$ and $h=1$, we have*

$$\varphi_{\alpha,0}(n,j)\equiv l_{\alpha}(n,j).$$

Note that the previous equivalence is analogous to the continuous case, see [7, Identity 32].

(ii) *Let $0<\alpha<1$. Taking $\lambda=0$ in Proposition 2.1.7-(iv), we have*

$$\sum_{j=0}^{\infty}\varphi^h_{\alpha,1-\alpha}(n-1,j)=1,\qquad n\in\mathbb{N}. \tag{2.6}$$

2.2 Fundamental solution

Here we investigate the representation of the solution to the fractional diffusion equation in discrete time and we prove several interesting properties related to it. We will initially remember the heat kernel in discrete time and its properties.

In [1], the authors defined the heat kernel in discrete time as

$$\mathcal{G}_n(x) := \frac{1}{\Gamma(n)} \int_0^\infty e^{-t} t^{n-1} G_t(x)\, dt, \quad n \in \mathbb{N},\ x \in \mathbb{R}^d \setminus \{0\}, \tag{2.7}$$

where $G_t(x)$ is the gaussian kernel defined by

$$G_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}, \qquad t > 0,\ x \in \mathbb{R}^d. \tag{2.8}$$

Remark 2.2.1. *Note that the authors in [1] considered the sequence (2.7) as the Poisson transform of the Gaussian kernel*

$$\mathcal{G}_n(x) = \mathcal{P}(G.(x))(n-1).$$

Thus, they used some properties of the Gaussian kernel to generate new results that fit perfectly in the discrete concepts. For example, $\mathcal{G}_n(x)$ satisfies the discrete semigroup property with respect to the time variable $(\mathcal{G}_n \circledast \mathcal{G}_m)(x) = \mathcal{G}_{n+m}(x)$, where $\circledast$ denotes the classical convolution on $\mathbb{R}^d$.

Let us recall that the Fourier transform of a function $u : \mathbb{R}^d \to \mathbb{R}$ is given by

$$\widehat{u}(\xi) = \mathcal{F}(u)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x) dx$$

and

$$\mathcal{F}^{-1}(u)(\xi) := \mathcal{F}(u)(-\xi)$$

denotes inverse Fourier transform of u.

The following properties of the heat kernel in discrete time (2.7) were proved in [1].

Proposition 2.2.2. *The heat kernel in discrete time $\mathcal{G}_n$ satisfies:*

(i) $\mathcal{G}_n(x) > 0, \quad n \in \mathbb{N},\, x \in \mathbb{R}^d \setminus \{0\}.$

(ii) $\displaystyle\int_{\mathbb{R}^d} \mathcal{G}_n(x)\; dx = 1.$

(iii) $\displaystyle\mathcal{F}(\mathcal{G}_n)(\xi) = \frac{1}{(1+|\xi|^2)^n}, \quad \xi \in \mathbb{R}^d.$

(iv) $\nabla \mathcal{G}_n(x) = \boldsymbol{\Delta}_x \mathcal{G}_n(x), \quad n \geq 2,\ x \in \mathbb{R}^d \setminus \{0\}.$

On the other hand, given a function f defined on $\mathbb{R}^d$, it easy to show that the function

$$w(n,x) := (1-\boldsymbol{\Delta_x})^{-n} f(x), \quad n \in \mathbb{N}_0,\ x \in \mathbb{R}^d,$$

is a solution of the following problem (see [1, Section 2]),

$$\begin{cases} \nabla w(n,x) = \boldsymbol{\Delta}_x w(n,x), & n \in \mathbb{N},\, x \in \mathbb{R}^d \setminus \{0\}, \\ w(0,x) = f(x). \end{cases} \tag{2.9}$$

From semigroup theory (see [36, Corollary 1.11]), the following identity holds

$$w(n,x) = (1-\boldsymbol{\Delta_x})^{-n} f(x) = \int_{\mathbb{R}^d} \mathcal{G}_n(x-y) f(y) dy := (\mathcal{G}_n \circledast f)(x),\ n \in \mathbb{N},\ x \in \mathbb{R}^d.$$

Observe that the total mass of w and the first moment are conservative in the discrete time n, that is,

$$\int_{\mathbb{R}^d} w(n,x)\, dx = \int_{\mathbb{R}^d} f(x)\, dx, \tag{2.10}$$

and

$$\int_{\mathbb{R}^d} x w(n,x)\, dx = \int_{\mathbb{R}^d} x f(x)\, dx, \tag{2.11}$$

respectively. As in the continuous case, the second moment is non-conservative.

$$\int_{\mathbb{R}^d} |x|^2\, w(n,x)\, dx = \int_{\mathbb{R}^d} |x|^2\, f(x)\, dx + 2dn.$$

For more details, see [1, Remark 2.6].

Once we remember the heat kernel in discrete time and its relationship with the homogeneous discrete in time heat initial value problem, we proceed to establish that the heat kernel in discrete time together to the discrete scaled Wright function induce a representation formula of solution for the following problem.

Let $0 < \alpha < 1$. Consider the fractional diffusion equation in discrete time, given by

$$\begin{cases} {}_C\nabla^\alpha u(n,x) = \mathbf{\Delta}_x u(n,x), \quad n \in \mathbb{N},\, x \in \mathbb{R}^d, \\ \\ u(0,x) = f(x), \end{cases} \tag{2.12}$$

where u and f are function defined on $\mathbb{N}_0 \times \mathbb{R}^d$ and $\mathbb{R}^d$ respectively.

According to the authors of the article [53], the time fractional diffusion equations are related to a class of Montroll–Weiss continuous time random walk (CTRW) models. Further, have become one of the standard physics approaches to model anomalous diffusion processes [29, 47, 65].

Let us define the fundamental solution

$$\mathcal{G}_n^\alpha(x) := \sum_{j=1}^{\infty} \varphi_{\alpha,1-\alpha}(n-1, j-1)\mathcal{G}_j(x), \quad n \in \mathbb{N},\, x \in \mathbb{R}^d \setminus \{0\}, \tag{2.13}$$

where the functions $\mathcal{G}_n(x)$ denote the heat kernel in discrete time defined by (2.7).

The next result shows that $(\mathcal{G}_n^\alpha \circledast f)(x)$ is the solution of (2.12). Notice that, like the solution fundamental of continuous-time fractional equation (see [53]), $\mathcal{G}_n^\alpha$ is not defined in zero. Thus, given an initial condition we have the solution for $n \in \mathbb{N}$.

Theorem 2.2.3. *Let f be a function on $L^p(\mathbb{R}^d)$. For $0 < \alpha < 1$, the function*

$$u(n,x) := (\mathcal{G}_n^\alpha \circledast f)(x),\ n \in \mathbb{N},\ x \in \mathbb{R}^d, \tag{2.14}$$

is the solution of the fractional diffusion equation in discrete time (2.12) on the Lebesgue $L^p(\mathbb{R}^d)$ spaces.

Proof. First of all, note that by Proposition 2.2.2-(ii) and (2.6), we can conclude that

$$\int_{\mathbb{R}^d} \mathcal{G}_n^\alpha(x)\, dx = 1. \tag{2.15}$$

Consequently, by Young's inequality for convolutions we have

$$\|u(n,\cdot)\|_p \leq \|f\|_p.$$

Now we see that u satisfies (2.12). Equation (2.9) implies

$$\begin{aligned}
\boldsymbol{\Delta}_x u(n,x) &= \sum_{j=1}^{\infty} \varphi_{\alpha,1-\alpha}(n-1,j-1)\nabla(\mathcal{G}_j \circledast f)(x) \\
&= \sum_{j=1}^{\infty} \varphi_{\alpha,1-\alpha}(n-1,j-1)(\mathcal{G}_j \circledast f)(x) - \sum_{j=1}^{\infty} \varphi_{\alpha,1-\alpha}(n-1,j-1)(\mathcal{G}_{j-1} \circledast f)(x) \\
&= \sum_{j=1}^{\infty} \varphi_{\alpha,1-\alpha}(n-1,j-1)(\mathcal{G}_j \circledast f)(x) - \sum_{j=1}^{\infty} \varphi_{\alpha,1-\alpha}(n-1,j)(\mathcal{G}_j \circledast f)(x) \\
&\qquad - \varphi_{\alpha,1-\alpha}(n-1,0)f(x).
\end{aligned}$$

Now, by Proposition 2.1.7-(ii) and (1.15), we have

$$\begin{aligned}
\varphi_{\alpha,1-\alpha}(n-1,0)f(x) &= \sum_{i=0}^{n-1} k^{1-\alpha}(n-1-i)\varphi_{\alpha,0}(i,0)f(x) \\
&= \sum_{i=0}^{n-1} k^{1-\alpha}(n-1-i)k^0(i)f(x) \\
&= k^{1-\alpha}(n-1)f(x).
\end{aligned}$$

Then, by (v) of Proposition 2.1.7, we get

$$\begin{aligned}
\boldsymbol{\Delta}_x u(n,x) &= \sum_{j=1}^{\infty} \varphi_{\alpha,1-\alpha}(n-1,j-1)(\mathcal{G}_j \circledast f)(x) - \sum_{j=1}^{\infty} \varphi_{\alpha,1-\alpha}(n-1,j)(\mathcal{G}_j \circledast f)(x) \\
&\qquad - k^{1-\alpha}(n-1)f(x) \\
&= \sum_{j=0}^{\infty} \left(\varphi_{\alpha,1-\alpha}(n-1,j) - \varphi_{\alpha,1-\alpha}(n-1,j+1)\right)(\mathcal{G}_{j+1} \circledast f)(x) \\
&\qquad - k^{1-\alpha}(n-1)f(x) \\
&= \sum_{j=0}^{\infty} \varphi_{\alpha,1-2\alpha}(n-1,j)(\mathcal{G}_{j+1} \circledast f)(x) - k^{1-\alpha}(n-1)f(x).
\end{aligned}$$

By the previous identity and (ii) of Proposition 2.1.7, we have that

$$\begin{aligned}
\boldsymbol{\Delta}_x \sum_{w=1}^{n} k^{\alpha}(n-w)u(w,x) &= \boldsymbol{\Delta}_x \sum_{w=0}^{n-1} k^{\alpha}(n-1-w)u(w+1,x) \\
&= \sum_{w=0}^{n-1} k^{\alpha}(n-1-w) \sum_{j=0}^{\infty} \varphi_{\alpha,1-2\alpha}(w,j)(\mathcal{G}_{j+1} \circledast f)(x) \\
&\quad - \sum_{w=0}^{n-1} k^{\alpha}(n-1-w)k^{1-\alpha}(w)f(x) \\
&= \sum_{w=0}^{n-1} k^{\alpha}(n-1-w) \sum_{j=0}^{\infty} \sum_{p=0}^{w} k^{1-2\alpha}(w-p)\varphi_{\alpha,0}(p,j)(\mathcal{G}_{j+1} \circledast f)(x) - f(x) \\
&= \sum_{j=0}^{\infty} \sum_{w=0}^{n-1} k^{\alpha}(n-1-w) \sum_{p=0}^{w} k^{1-2\alpha}(w-p)\varphi_{\alpha,0}(p,j)(\mathcal{G}_{j+1} \circledast f)(x) - f(x) \\
&= \sum_{j=0}^{\infty} \sum_{p=0}^{n-1} k^{1-\alpha}(n-1-p)\varphi_{\alpha,0}(p,j)(\mathcal{G}_{j+1} \circledast f)(x) - f(x) \\
&= \sum_{j=0}^{\infty} \varphi_{\alpha,1-\alpha}(n-1,j)(\mathcal{G}_{j+1} \circledast f)(x) - f(x) \\
&= u(n,x) - f(x),
\end{aligned}$$

that is,

$$u(n,x) = \boldsymbol{\Delta}_x \sum_{w=1}^{n} k^{\alpha}(n-w)u(w,x) + f(x).$$

Now, convolving the above identity by $k^{1-\alpha}$, we obtain

$$\begin{aligned}
\sum_{j=0}^{n} k^{1-\alpha}(n-j)u(j,x) &= \boldsymbol{\Delta}_x \sum_{j=0}^{n} u(j,x) - \boldsymbol{\Delta}_x u(0,x) + k^{2-\alpha}(n)f(x) \\
&= \boldsymbol{\Delta}_x \sum_{j=1}^{n} u(j,x) + k^{2-\alpha}(n)f(x).
\end{aligned}$$

By Proposition 1.2.1-(iv), we can conclude that

$$\begin{aligned}
\sum_{j=0}^{n} k^{1-\alpha}(n-j)u(j,x) - &\sum_{j=0}^{n-1} k^{1-\alpha}(n-1-j)u(j,x) \\
&= \boldsymbol{\Delta}_x u(n,x) + k^{1-\alpha}(n)f(x).
\end{aligned}$$

Hence, the result follows from

$$\sum_{j=0}^{n} k^{1-\alpha}(n-j)u(j,x) - \sum_{j=0}^{n-1} k^{1-\alpha}(n-1-j)u(j,x) - k^{1-\alpha}(n)f(x) = {}_C\nabla^{\alpha}u(n,x).$$

Finally, the uniqueness follows from we can construct the solution of (2.12) by successive calculations. That is, it is defined by a law of recurrence. In fact, given any value $f(x)$, there exists a unique solution u, with values $u(n,x)$ for all $n \in \mathbb{N}$, since if there is another solution that verifies the initial condition, then they coincide in all points since the recurrence law itself determines the subsequent values of the solution. $\square$

In the following results we show other representations for $\mathcal{G}_n^{\alpha}$. In the first result, we represent $\mathcal{G}_n^{\alpha}(x)$ using the Poisson transform of the Gaussian function, while in the second one we use the Fox H-function. This fact in turn gives other representations of the solution (2.14).

Proposition 2.2.4. *Let* $0 < \alpha < 1$. *Then,* (2.13) *is equivalent to*

$$\mathcal{G}_n^{\alpha}(x) = \int_0^{\infty}\int_0^{\infty} e^{-s}\frac{s^{n-1}}{(n-1)!}\psi_{\alpha,1-\alpha}(s,t)G_t(x)\, ds\, dt, \quad n \in \mathbb{N},\ x \in \mathbb{R}^d \setminus \{0\}, \tag{2.16}$$

where G_t *is the Gaussian kernel defined by* (2.8) *and* $\psi_{\alpha,\beta}$ *is defined by* (1.8).

Proof. From Proposition 2.1.7-(*iii*), we get

$$\begin{aligned}
\mathcal{G}_n^{\alpha}(x) &= \sum_{j=1}^{\infty} \varphi_{\alpha,1-\alpha}(n-1,j-1)\mathcal{G}_j(x) \\
&= \sum_{j=1}^{\infty} \varphi_{\alpha,1-\alpha}(n-1,j-1)\int_0^{\infty} e^{-t}\frac{t^{j-1}}{(j-1)!}G_t(x)\, dt \\
&= \int_0^{\infty} e^{-t}\sum_{j=1}^{\infty} \varphi_{\alpha,1-\alpha}(n-1,j-1)\frac{t^{j-1}}{(j-1)!}G_t(x)\, dt \\
&= \int_0^{\infty}\int_0^{\infty} e^{-s}\frac{s^{n-1}}{(n-1)!}\psi_{\alpha,1-\alpha}(s,t)G_t(x)\, ds\, dt.
\end{aligned}$$

The result follows. $\square$

Proposition 2.2.5. *Let $0 < \alpha < 1$. Then*

$$\mathcal{G}_n^\alpha(x) = \frac{1}{\Gamma(n)\pi^{d/2}|x|^d} H_{13}^{30}\left[\frac{|x|^2}{4} \middle| \begin{array}{c} (1,\alpha) \\ (n,\alpha),(\frac{d}{2},1),(1,1) \end{array} \right],$$

where H_{31}^{03} denotes the Fox H-function.

Proof. By [23, Theorem 3.1], [7, Theorem 15-(*ii*).] and [53, Theorem 2.12], we have the following subordination formula

$$\frac{1}{\pi^{d/2}|x|^d} H_{21}^{02}\left[\frac{4t^\alpha}{|x|^2} \middle| \begin{array}{c} (1-\frac{d}{2},1),(0,1) \\ (0,\alpha) \end{array} \right] = \int_0^\infty \psi_{\alpha,1-\alpha}(t,s) G_s(x)\, ds.$$

Now,

$$\begin{aligned}
\mathcal{G}_n^\alpha(x) &= \frac{1}{\Gamma(n)\pi^{d/2}|x|^d} \int_0^\infty e^{-t} t^{n-1} H_{32}^{12}\left[\frac{4t^\alpha}{|x|^2} \middle| \begin{array}{c} (1-\frac{d}{2},1),\ (0,1),\ (0,1) \\ (0,1),\ (0,\alpha) \end{array} \right] dt \\
&= \frac{1}{\Gamma(n)\pi^{d/2}|x|^d} H_{42}^{13}\left[\frac{4}{|x|^2} \middle| \begin{array}{c} (1-n,\alpha),\ (1-\frac{d}{2},1),\ (0,1),\ (0,1) \\ (0,1),\ (0,\alpha) \end{array} \right] \\
&= \frac{1}{\Gamma(n)\pi^{d/2}|x|^d} H_{31}^{03}\left[\frac{4}{|x|^2} \middle| \begin{array}{c} (1-n,\alpha),(1-\frac{d}{2},1),(0,1) \\ (0,\alpha) \end{array} \right] \\
&= \frac{1}{\Gamma(n)\pi^{d/2}|x|^d} H_{13}^{30}\left[\frac{|x|^2}{4} \middle| \begin{array}{c} (1,\alpha) \\ (n,\alpha),(\frac{d}{2},1),(1,1) \end{array} \right],
\end{aligned}$$

where we have used [54, Corollary 2.3.1], (1.11) and (1.12). ☐

The following proposition states some basic properties of the fundamental solution.

Proposition 2.2.6. *Let $x \in \mathbb{R}^d \setminus \{0\}$. The function $\mathcal{G}_n^\alpha$ satisfies:*

(*i*) $\mathcal{G}_n^\alpha(x) > 0, \quad n \in \mathbb{N}$.

(*ii*) $\displaystyle\int_{\mathbb{R}^d} \mathcal{G}_n^\alpha(x)\, dx = 1$.

(*iii*) $\mathcal{F}(\mathcal{G}_n^\alpha)(\xi) = \mathcal{E}_{\alpha,1}(-|\xi|^2, n), \quad \xi \in \mathbb{R}^d$.

(*iv*) $\int_{\mathbb{R}^d} |x|^2 \mathcal{G}_n^\alpha(x)\,dx = 2k^{\alpha+1}(n-1)d.$

Proof. (i) follows from (vi) of Proposition 2.1.7 and (i) of the Proposition 2.2.2. (ii) was showed in the proof of Theorem 2.2.3 (see (2.15)). Next, let us prove the item (iii). Since $\mathcal{F}(G_t)(\xi) = e^{-t|\xi|^2}$, for $\xi \in \mathbb{R}^d$, it follows from (1.9) that

$$\int_0^\infty \psi_{\alpha,1-\alpha}(s,t)e^{-t|\xi|^2}\,dt = E_{\alpha,1}(-|\xi|^2 s^\alpha).$$

Theorem 1.2.2 implies that

$$\mathcal{F}(\mathcal{G}_{n,h}^\alpha)(\xi) = \int_0^\infty e^{-s}\frac{s^{n-1}}{\Gamma(n)}E_{\alpha,1}(-|\xi|^2 s^\alpha)\,ds = \mathcal{E}_{\alpha,1}(-|\xi|^2, n).$$

Finally, it is known that

$$\int_{\mathbb{R}^d} |x|^2 G_t(x)\,dx = 2\,d\,t.$$

Then, by Fubini's Theorem and (1.10), we have that

$$\begin{aligned}\int_{\mathbb{R}^d} |x|^2\mathcal{G}_n^\alpha(x)\,dx =& 2d\int_0^\infty\int_0^\infty e^{-s}\frac{s^{n-1}}{(n-1)!}\psi_{\alpha,1-\alpha}(s,t)t\,dt\,ds\\ =&2d\int_0^\infty e^{-s}\frac{s^{n-1}}{(n-1)!}g_{\alpha+1}(s)\,ds\\ =&2dk^{\alpha+1}(n-1).\end{aligned}$$

Thus, we get item (iv). $\square$

Remark 2.2.7. *We have that the total mass of solution of* (2.12)*, given by*

$$u(n,x) = \sum_{j=1}^\infty \varphi_{\alpha,1-\alpha}(n-1,j-1)(\mathcal{G}_j \circledast f)(x)$$

is conservative. Indeed,

$$\begin{aligned}\int_{\mathbb{R}^d} u(n,x)\,dx &= \sum_{j=1}^\infty \varphi_{\alpha,1-\alpha}(n-1,j-1)\int_{\mathbb{R}^d}(\mathcal{G}_j \circledast f)(x)\,dx\\ &= \int_{\mathbb{R}^d} f(x)\,dx,\end{aligned}$$

where in the last equality we have used (2.10) and (2.6). This fact leads us to think that the total mass of solutions should have importance in the asymptotic behavior of solutions. On the other hand, the first moment is also conservative:

$$\begin{aligned}\int_{\mathbb{R}^d} x\,u(n,x)dx &= \sum_{j=1}^{\infty} \varphi_{\alpha,1-\alpha}(n-1,j-1)\int_{\mathbb{R}^d} x\,(\mathcal{G}_j * f)(x)\,dx \\ &= \int_{\mathbb{R}^d} x f(x)\,dx,\end{aligned}$$

as long as $(1+|x|)f \in L^1(\mathbb{R}^d)$ *(see (2.11)). However, in the same way that* w*, the second moment of* u *is not conserved in time. In fact,*

$$\begin{aligned}\int_{\mathbb{R}^d} |x|^2\,u(n,x)dx &= \int_{\mathbb{R}^d} |x|^2\,f(x)dx + 2dh\sum_{j=1}^{\infty} \varphi^h_{\alpha,1-\alpha}(n-1,j-1)j \\ &= \int_{\mathbb{R}^d} |x|^2\,f(x)dx + 2\Gamma(2)dk^{\alpha+1}(n-1),\end{aligned}$$

where used (ix) *of Proposition 2.1.7.*

2.3 Asymptotic decay and asymptotic behavior of solution

Now we will present the asymptotic decay of the solution of (2.12) (which is given by (2.14)) in $L^p(\mathbb{R}^d)$ spaces and the corresponding large-time behaviour. Initially, we show the following estimates of the fundamental solution $\mathcal{G}_n^\alpha$ in $L^p(\mathbb{R}^d)$-spaces and we state $L^p(\mathbb{R}^d)$-estimates for $\boldsymbol{\nabla}_x\mathcal{G}_n^\alpha(x)$.

Lemma 2.3.1. *Let* $0<\alpha<1$*. Then there exists* $C_p>0$ *such that*

$$\|\mathcal{G}_n^\alpha\|_p \le C_p\frac{1}{n^{\frac{\alpha d}{2}(1-1/p)}}, \quad n\in\mathbb{N},$$

for $p\in[1,\infty]$ *if* $d=1$*, for* $p\in[1,\infty)$ *if* $d=2$*, and for* $p\in[1,\frac{d}{d-2})$ *if* $d>2$*.*

Proof. It is well known (see [42, p.334 (3.326)]) that there exists C_p (independent of t) such that $||G_t||_p = C_p\frac{1}{t^{\frac{d}{2}(1-\frac{1}{p})}}$. Then for n large enough and the values of p

given in the hypothesis, by (2.16) and (1.10) one gets

$$
\begin{aligned}
\|\mathcal{G}_n^\alpha\|_p &\leq \int_0^\infty \int_0^\infty e^{-s}\frac{s^{n-1}}{(n-1)!}\psi_{\alpha,1-\alpha}(s,t)||G_t||_p \, ds \, dt \\
&\leq \frac{C_p}{\Gamma(n)}\int_0^\infty \int_0^\infty e^{-t}t^{n-1}\psi_{\alpha,1-\alpha}(t,s)s^{\frac{d}{2}(\frac{1}{p}-1)+1-1}\, dt \, ds \\
&\leq \frac{C_p}{\Gamma(n)}\int_0^\infty e^{-t}t^{n-\alpha\frac{d}{2}(1-\frac{1}{p})-1}\, dt \\
&= C_p\frac{\Gamma(n-\alpha\frac{d}{2}(1-\frac{1}{p}))}{\Gamma(n)} \\
&\leq \frac{C_p}{n^{\alpha\frac{d}{2}(1-\frac{1}{p})}},
\end{aligned}
$$

where we have applied the asymptotic behaviour of the Gamma function (1.1). Since the function $\mathcal{G}_n^\alpha$ belongs to $L^p(\mathbb{R}^d)$ for all $n \in \mathbb{N}$, then the result is valid for all $n \in \mathbb{N}$. □

We state $L^p(\mathbb{R}^d)$-estimates for $\nabla_x \mathcal{G}_n^\alpha(x)$, which are useful for study the large-time behaviour of the solution (2.14).

Lemma 2.3.2. *Let $0 < \alpha < 1$. Then there exists $C_p > 0$ such that*

$$\|\nabla_x \mathcal{G}_n^\alpha\|_p \leq C_p \frac{1}{n^{\frac{\alpha d}{2}(1-1/p)+\frac{\alpha}{2}}}, \quad n \in \mathbb{N},$$

for $p \in [1,\infty)$ if $d = 1$, and for $p \in [1, \frac{d}{d-1})$ if $d > 1$.

Proof. The proof is similar to the proof of Lemma 2.3.1 by use of

$$\|\nabla_x G_t\|_p = C_p \frac{1}{t^{\frac{d}{2}(1-\frac{1}{p})+1/2}},$$

see [42, p.334 (3.326)]. □

Next, let us present a result about the $L^p(\mathbb{R}^d) - L^q(\mathbb{R}^d)$ asymptotic decay for u.

Theorem 2.3.3. *Let $1 \leq q \leq p \leq \infty$. If $f \in L^q(\mathbb{R}^d)$, then the solution u of (2.12) satisfies* *(i) If* $q = \infty$*, then* $\|u(n,\cdot)\|_\infty \le \|f\|_\infty$.

(ii) If $1 \le q < \infty$ *and* $d > 2q$*, then for each* $p \in [q, \frac{dq}{d-2q})$

$$\|u(n,\cdot)\|_p \le C_p \frac{1}{n^{\frac{\alpha d}{2}(1/q-1/p)}} \|f\|_q. \tag{2.17}$$

(iii) If $1 \le q < \infty$ *and* $d = 2q$*, then for each* $p \in [q, \infty)$ *the estimate* (2.17) *holds.*

(iv) If $1 \le q < \infty$ *and* $d < 2q$*, then for each* $p \in [q, \infty]$ *the estimate* (2.17) *holds.*

Here, C_p *is a constant independent of* n.

Proof. Take $r \ge 1$ such that $1 + 1/p = 1/q + 1/r$, and applying Young's inequality we get

$$\|u(n,\cdot)\|_p = \|\mathcal{G}_n^\alpha \circledast f\|_p \le \|\mathcal{G}_n^\alpha\|_r \|f\|_q.$$

Now, we apply Lemma 2.3.1 to estimate $\|\mathcal{G}_n^\alpha\|_r$. For the case (i), if $q = \infty$, then $p = \infty, r = 1$, and therefore since $\|\mathcal{G}_n^\alpha\|_1 = 1$, the result follows. Note that in the case (ii), if $1 \le q < \infty$ and $d > 2q$, then the condition $q \le p < \frac{dq}{d-2q}$ implies $1 \le r < \frac{d}{d-2}$. So, by Lemma 2.3.1 we get the desired estimates. The cases (iii) and (iv) follow in a similar way. □

In this part we study the asymptotic behaviour of solution u of problem given by (2.12). Suppose $f \in L^1(\mathbb{R}^d)$, set

$$M := \int_{\mathbb{R}^d} f(x)\, dx.$$

Before to show the main result of this section, we need the following decomposition lemma (see [34]).

Lemma 2.3.4. *Suppose* $f \in L^1(\mathbb{R}^d)$ *such that* $\int_{\mathbb{R}^d} |x||f(x)|dx < \infty$*. Then there exists* $F \in L^1(\mathbb{R}^d; \mathbb{R}^d)$ *such that*

$$f = \left(\int_{\mathbb{R}^d} f(x)dx\right)\delta_0 + div\, F$$

in the distributional sense and

$$\|F\|_{L^1(\mathbb{R}^d;\mathbb{R}^d)} \leq C_d \int_{\mathbb{R}^d} |x||f(x)|dx.$$

The following estimate shows that the difference on $L^p(\mathbb{R}^d)$ between the solution $u(n,x)$ and $M\mathcal{G}_n^\alpha(x)$ decays to zero like $o\left(1/n^{\frac{\alpha d}{2}(1-\frac{1}{p})}\right)$ as n goes to infinity. Moreover, if $|x|f \in L^1(\mathbb{R}^d)$, we are able to improve the convergence.

Theorem 2.3.5. *Let* $1 \leq p \leq \infty$ *and* u *be the solution of* (2.12).

(*i*) *Then*

$$n^{\frac{\alpha d}{2}(1-\frac{1}{p})}\|u(n,\cdot) - M\mathcal{G}_n^\alpha\|_p \to 0, \quad as \quad n \to \infty,$$

for $p \in [1,\infty)$ *if* $d = 1$, *and for* $p \in [1, \frac{d}{d-1})$ *if* $d > 1$,

(*ii*) *Suppose in addition that* $|x|f \in L^1(\mathbb{R}^d)$, *then*

$$n^{\frac{\alpha d}{2}(1-\frac{1}{p})}\|u(n,\cdot) - M\mathcal{G}_n^\alpha\|_p \lesssim n^{-\alpha/2},$$

for $p \in [1,\infty)$ *if* $d = 1$, *and for* $p \in [1, \frac{d}{d-1})$ *if* $d > 1$.

Proof. First, we prove assertion (*ii*). Since $f, |x|f \in L^1(\mathbb{R}^d)$, by decomposition Lemma 2.3.4 there exists $\psi \in L^1(\mathbb{R}^d;\mathbb{R}^d)$ such that

$$\begin{aligned} u(n,x) &= (\mathcal{G}_n^\alpha \circledast (M\delta_0 + \operatorname{div}\psi(\cdot)))(x) \\ &= M\mathcal{G}_n^\alpha(x) + (\boldsymbol{\nabla}_x\mathcal{G}_n^\alpha \circledast \psi)(x), \end{aligned}$$

in the distributional sense, and

$$\|\psi\|_1 \leq C_d\||x|f\|_1 < \infty.$$

Consequently,

$$\|u(n,\cdot) - M\mathcal{G}_n^\alpha\|_p \leq C_d\|\boldsymbol{\nabla}_x\mathcal{G}_n^\alpha\|_p\||x|f\|_1 \leq C_{p,f}\frac{1}{n^{\frac{\alpha d}{2}(1-1/p)+\frac{\alpha}{2}}}, \tag{2.18}$$

Hence the assertion (ii) is proved.

To prove (i), we choose a sequence $(\eta_j) \subset C_0^\infty(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \eta_j(x)\,dx = M$ for all j, and $\eta_j \to f$ in $L^1(\mathbb{R}^d)$. For each j, by Lemma 2.3.1 and (2.18), we get

$$\begin{aligned} \|u(n,\cdot) - M\mathcal{G}_n^\alpha\|_p &\leq \|\mathcal{G}_n^\alpha \circledast (f-\eta_j)\|_p + \|\mathcal{G}_n^\alpha \circledast \eta_j - M\mathcal{G}_n^\alpha\|_p \\ &\leq \|\mathcal{G}_n^\alpha\|_p \|f-\eta_j\|_1 + \|\mathcal{G}_n^\alpha \circledast \eta_j - M\mathcal{G}_n^\alpha\|_p \\ &\leq C_p \frac{1}{n^{\frac{\alpha d}{2}(1-1/p)}} \|f-\eta_j\|_1 + C_{p,\eta_j} \frac{1}{n^{\frac{\alpha d}{2}(1-1/p)+\frac{\alpha}{2}}}. \end{aligned}$$

Then

$$\limsup_{n\to\infty} n^{\frac{\alpha d}{2}(1-1/p)} \|u(n,\cdot) - M\mathcal{G}_n^\alpha\|_p \leq C_p \|f-\eta_j\|_1.$$

The assertion follows by letting $j \to \infty$. $\square$

3. Subordination principle and the discrete Hilfer fractional operator

In this chapter, using the discrete scaled Wright function, a subordination principle is proved. This principle relates a sequence of solution operators, given by a discrete C-semigroup, for the abstract Cauchy problem of first order in discrete-time, with a sequence of solution operators for the abstract Cauchy problem of fractional order $0 < \alpha < 1$ in discrete-time. As an application, we establish the explicit solution of the abstract Cauchy problem in discrete-time that involves the Hilfer fractional difference operator and prove that, in some cases, such solution converges to zero.

3.1 Subordination principle

Initially, we introduce the notion of discrete C-semigroup and present some interesting properties. Moreover, we introduce the notion of (α, ν)-resolvent sequences.

Let C be a bounded and injective operator defined on X. Now, suppose that a strongly continuous operator-valued sequence $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ satisfies the following conditions:

(i) $\mathcal{T}(0) = C$.

(ii) $C\mathcal{T}(n+m) = \mathcal{T}(n)\mathcal{T}(m)$ for $n, m \in \mathbb{N}$.

In analogy with the continuous case [33], we say that the family $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0}$ is a *discrete C-semigroup.*

From the definition, it is clear that $\mathcal{T}(n)$ commutes with C. Moreover, by simple iteration, we found that any discrete C-semigroup have the form

$$\mathcal{T}(n) = [C^{-1}\mathcal{T}(1)]^n C = C^{-(n-1)}\mathcal{T}(1)^n, \quad n \in \mathbb{N}_0. \tag{3.1}$$

From now on, A will denote a closed linear operator with domain $D(A)$ defined on X and $\rho(A)$ will denote its resolvent set.

Proposition 3.1.1. *Let $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ be strongly continuous and satisfying the following properties;*

(i) $\mathcal{T}(n)x \in D(A)$ *for all* $x \in X$.

(ii) $A\mathcal{T}(n)x = \mathcal{T}(n)Ax$ *for each* $x \in D(A)$ *and* $n \in \mathbb{N}_0$.

(iii) $\mathcal{T}(n)x = x + A\sum_{j=0}^{n}\mathcal{T}(j)x$, *for all* $n \in \mathbb{N}_0$ *and* $x \in X$.

Then, $1 \in \rho(A)$ *and* $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ *is a discrete C-semigroup with* $C := (I - A)^{-1}$.

Proof. With $n = 0$ the property (iii) gives

$$\mathcal{T}(0)x = x + A\mathcal{T}(0)x, \quad x \in X,$$

which, together with (ii), implies that $1 \in \rho(A)$ and $T(0) = (I - A)^{-1}$. Using the identity

$$(I - A)^{-1} - I = A(I - A)^{-1},$$

we have that for all $x \in X$

$$\begin{aligned}\mathcal{T}(1)x =& x + A\mathcal{T}(0)x + A\mathcal{T}(1)x \\ =& \mathcal{T}(0)x + A\mathcal{T}(1)x,\end{aligned}$$

or

$$\mathcal{T}(1)x = (I - A)^{-2}x.$$

Iterating (iii) we find, that for all $n \in \mathbb{N}_0$,

$$\mathcal{T}(n)x = (I - A)^{-(n+1)}x = (I - A)^{-n}Cx = [C^{-1}\mathcal{T}(1)]^n Cx, \tag{3.2}$$

which proves that $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ is a C-semigroup with $C := (I - A)^{-1}$. $\square$

Definition 3.1.2. *We say that the family $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ satisfying (i)–(iii) in the above proposition is a discrete C-semigroup generated by A.*

The following result shows an interesting new interpretation of one of the main results in the reference [56]. The striking point that shows the next theorem is that, in strong contrast with the continuous case, the natural family of operators behind of the well posedness of the discrete abstract Cauchy problem of first order is a discrete C-semigroup instead of a discrete semigroup, which was the first attempt in [56].

Theorem 3.1.3. *Let $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ be a discrete C-semigroup generated by A. Then the discrete-time abstract Cauchy problem of first order*

$$\Delta u(n) = Au(n+1), \quad n \in \mathbb{N}_0,$$

with initial condition $u(0) = u_0 \in X$ admits the solution

$$u(n) = C^{-1}\mathcal{T}(n)u_0.$$

Proof. Note that $k^1(n) = 1$ and $\Delta k^1(n) = 0$ for all $n \in \mathbb{N}_0$. Then,

$$\begin{aligned}\Delta\mathcal{T}(n)x &= A\Delta\sum_{j=0}^{n}\mathcal{T}(j)x \\ &= A\left(\sum_{j=0}^{n+1}\mathcal{T}(j)x - \sum_{j=0}^{n}\mathcal{T}(j)x\right) \\ &= A\mathcal{T}(n+1)x.\end{aligned}$$

We define $u(n) := C^{-1}\mathcal{T}(n)u_0$. Since $\mathcal{T}(n)x \in D(A)$ for all $x \in X$ and $n \in \mathbb{N}_0$, we obtain $u(n) \in D(A)$ for all $n \in \mathbb{N}_0$. From the above identity, it is clear that $\Delta u(n) = Au(n+1)$. Finally, since $T(0) = C$, we obtain $u(0) = u_0$. $\square$

Motivated by Proposition 3.1.1, we now introduce the following sequence of bounded and linear operators that is the discrete counterpart of the concept of resolvent families of operators for fractional evolution equations in continuous time. See [57] for a recent review of this concept and its main properties, and [58] for their application to nonlinear fractional evolution equations in the setting of Banach spaces.

In what follows, the symbol $*$ denotes the discrete convolution of two sequences $f, g \in s(\mathbb{N}_0, X)$ defined by

$$(f * g)(n) := \sum_{j=0}^{n} f(n-j)g(j).$$

Definition 3.1.4. *Let $\alpha, \nu > 0$ be given. An operator-valued sequence*

$$\{S_{\alpha,\nu}(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$$

is called a discrete (α, ν)-resolvent sequence generated by A if it satisfies the following conditions:

(i) $S_{\alpha,\nu}(n)x \in D(A)$ for all $x \in X$ and $S_{\alpha,\nu}(n)Ax = AS_{\alpha,\nu}(n)x$ for each $n \in \mathbb{N}_0$ and $x \in D(A)$.

(ii) $S_{\alpha,\nu}(n)x = k^{\nu}(n)x + A(k^{\alpha} * S_{\alpha,\nu})(n)x$ for all $n \in \mathbb{N}_0$ and each $x \in X$.

Note that $S_{1,1}(n) = \mathcal{T}(n)$ is the C-semigroup generated by A. The case $\alpha = \nu$ was introduced in [5, Definition 3.1] and used, among others, in [6, Section 2] and [45] in connection with linear and nonlinear fractional abstract difference equations. In particular, in [6, Proposition 2.2] it was proved that if A is a bounded operator with norm less than 1, then the following representation holds:

$$S_{\alpha,\alpha}(n) = \sum_{j=0}^{\infty} k^{\alpha(j+1)}(n)A^j.$$

The following is the main result of this chapter and shows a striking relation between discrete (α,ν)-resolvent sequences and discrete C-semigroups. It allows a representation of a discrete (α,ν)-resolvent sequence in terms of the discrete C-semigroup generated by A. This result extends and improves [5, Theorem 3.2] and [6, Thorem 2.3].

Theorem 3.1.5. *Let $0<\alpha<1$, $\alpha\le\nu$ be given. Let $\{\mathcal{T}(n)\}_{n\in\mathbb{N}}$ be a discrete C-semigroup generated by A. Then the family*

$$S_{\alpha,\nu}(n)x=\sum_{j=0}^{\infty}\varphi_{\alpha,\nu-\alpha}(n,j)\mathcal{T}(j)x, \quad n\in\mathbb{N}_0, \tag{3.3}$$

is a discrete (α,ν)-resolvent sequence generated by A.

Proof. From definition of C-semigroup and the fact that A is closed, we have that $S_{\alpha,\nu}(n)x\in D(A)$ for all $x\in X$ and $S_{\alpha,\nu}(n)Ax=AS_{\alpha,\nu}(n)x$ for each $x\in D(A)$. The group property of k^α shows that

$$A(k^\alpha * S_{\alpha,\nu})(n)x=\sum_{j=0}^{\infty}\varphi_{\alpha,\nu}(n,j)A\mathcal{T}(j)x.$$

Note that (3.2) and the identity $A(I-A)^{-1}=(I-A)^{-1}-I$ imply that $ACx=C^{j+1}x-C^jx$ for all $x\in X$. Therefore,

$$\begin{aligned}A(k^\alpha * S_{\alpha,\nu})(n)x&=\sum_{j=0}^{\infty}\varphi_{\alpha,\nu}(n,j)[C^{j+1}-C^j]x\\&=\sum_{j=0}^{\infty}\varphi_{\alpha,\nu}(n,j)C^{j+1}x-\sum_{j=0}^{\infty}\varphi_{\alpha,\nu}(n,j)C^jx.\end{aligned}$$

Hence

$$\begin{aligned}A(k^\alpha * S_{\alpha,\nu})(n)x&=\sum_{j=0}^{\infty}\varphi_{\alpha,\nu}(n,j)C^{j+1}x-\sum_{j=1}^{\infty}\varphi_{\alpha,\nu}(n,j)C^jx-k^\nu(n)x\\&=\sum_{j=0}^{\infty}\varphi_{\alpha,\nu}(n,j)C^{j+1}x-\sum_{j=0}^{\infty}\varphi_{\alpha,\nu}(n,j+1)C^{j+1}x-k^\nu(n)x\\&=\sum_{j=0}^{\infty}(\varphi_{\alpha,\nu}(n,j)-\varphi_{\alpha,\nu}(n,j+1))C^{j+1}x-k^\nu(n)x,\end{aligned}$$

where in the first equality we have used (2.1.5) (for $j = 0$). Applying the Proposition 2.1.7, item (v), we get

$$A(k^\alpha * S_{\alpha,\nu})(n)x = \sum_{j=0}^{\infty} \varphi_{\alpha,\nu-\alpha}(n, j+1)C^{j+1}x - k^\nu(n)x = S_{\alpha,\nu}(n)x - k^\nu(n)x.$$

It proves that $\{S_{\alpha,\nu}(n)\}_{n\in\mathbb{N}_0}$ is a discrete (α, ν)-resolvent sequence generated by A. □

Corollary 3.1.6. *Let $0 < \alpha \le \varrho \le 1$ be given. Let A be a closed and linear operator defined on a Banach space X such that $1 \in \rho(A)$. Then the family*

$$S_{\alpha,\varrho}(n)x = \sum_{j=0}^{\infty} \varphi_{\alpha,\varrho-\alpha}(n, j)(I - A)^{-(j+1)}x, \quad n \in \mathbb{N}_0, \ x \in X, \tag{3.4}$$

is a discrete (α, ϱ)-resolvent sequence generated by A.

Proof. By hypothesis, $C := (I - A)^{-1}$ exists and the operator $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0}$ given by $\mathcal{T}(n) = (I - A)^{-(n+1)}$ is bounded on X. On the other hand,

$$\mathcal{T}(n) = (I - A)^{-(n+1)} = C^{-(n-1)}\mathcal{T}(1)^n.$$

Hence, the operator $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0}$ is a discrete C-semigroup. Thus, the result follows from Theorem 3.1.5. □

We recall that an operator-valued sequence $\{S(n)\}_{n\in\mathbb{N}_0} \in \mathcal{B}(X)$ is said to be *summable* if

$$\| S \|_{\ell_1} := \sum_{n=0}^{\infty} \| S(n) \| < \infty.$$

Theorem 3.1.7. *Let A be a closed linear operator defined on a Banach space X such that $1 \in \rho(A)$ and*

$$\|(I - A)^{-1}\| < 1. \tag{3.5}$$

Then A generates a summable discrete (α, α)-resolvent sequence $\{S_{\alpha,\alpha}(n)\}_{n\in\mathbb{N}_0}$.

Proof. Since $1 \in \rho(A)$, then by Corollary 3.1.6 the family

$$S_{\alpha,\alpha}(n)x = \sum_{j=0}^{\infty} \varphi_{\alpha,0}(n,j)(I-A)^{-(j+1)}x, \quad n \in \mathbb{N}_0, \ x \in X,$$

is a discrete (α,α)-resolvent sequence generated by A. We will prove that it is summable. Indeed, since $0 \le \varphi_{\alpha,0}(n,j) \le 1$ for $j \in \mathbb{N}_0$, then

$$\begin{aligned}\sum_{n=0}^{\infty} \| S_{\alpha,\alpha}(n) \| \le \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \varphi_{\alpha,0}(n,j) \| (I-A)^{-(j+1)} \| \le \sum_{j=0}^{\infty} \| (I-A)^{-(j+1)} \| \\ \le \|(I-A)^{-1}\| \sum_{j=0}^{\infty} \| (I-A)^{-1} \|^{j} < \infty.\end{aligned}$$

□

The following example provides concrete conditions on A under which (3.5) holds.

Example 3.1.8. *Let A be the generator of a C_0-semigroup strictly contractive. For instance, on $X := L^1(\mathbb{R})$ we define*

$$(T(t)f)(s) = \begin{cases} \beta f(t+s), & if \quad s \in [-t,0], \\ \\ f(t+s), & otherwise, \end{cases}$$

where $0 < \beta < 1$ is arbitrary. Then $T(t)$ is a C_0-semigroup and $\|T(t)\| = \beta < 1$ (since $\|T(t)\mathbb{1}_{[0,t]}\|_1 = \beta\|\mathbb{1}_{[0,t]}\|_1$).

We deduce that $1 \in \rho(A)$ and $\|(I-A)^{-1}\| < 1$. Indeed,

$$\|(I-A)^{-1}\| = \| \int_0^{\infty} e^{-t}T(t)\, dt\| \le \int_0^{\infty} e^{-t}\|T(t)\|dt < \beta < 1.$$

The last part of the earlier example shows the following result.

Corollary 3.1.9. *Let A be the generator of a C_0-semigroup strictly contractive, then $1 \in \rho(A)$ and $\|(I-A)^{-1}\| < 1$.*

The following consequence establishes an interesting link between the stability property of the C-semigroup with those of the (α,ν)-resolvent sequence, both generated by the same operator A.

Proposition 3.1.10. *Let $0<\alpha<1$ and $\alpha\leq\nu$. Assume that A is the generator of a discrete C-semigroup $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0}$ and there exist constants $M,\omega>0$ such that*

$$\parallel \mathcal{T}(n) \parallel\leq M(1+\omega)^{-(n+1)}, \quad for\ all\ n\in\mathbb{N}_0.$$

Then the discrete (α,ν)-resolvent sequence $\{S_{\alpha,\nu}(n)\}_{n\in\mathbb{N}_0}$ generated by A satisfies

$$\parallel S_{\alpha,\nu}(n) \parallel\leq M\mathcal{E}_{\alpha,\nu}(-\omega,n).$$

In particular, if $0<\nu<1$, then $\parallel S_{\alpha,\nu}(n)\parallel\to 0$ as $n\to\infty$.

Proof. Let $\nu\geq\alpha$. The sequence (3.3) together with Proposition 2.1.7 (iv) and (vii) can be used to obtain that

$$\begin{aligned}
\parallel S_{\alpha,\nu}(n) \parallel &= \left\|\sum_{j=0}^{\infty}\varphi_{\alpha,\nu-\alpha}(n,j)\mathcal{T}(j)\right\| \\
&\leq \sum_{j=0}^{\infty}\parallel \varphi_{\alpha,\nu-\alpha}(n,j)\mathcal{T}(j)\parallel \\
&= \sum_{j=0}^{\infty}\varphi_{\alpha,\nu-\alpha}(n,j)\parallel \mathcal{T}(j)\parallel \\
&\leq M\sum_{j=0}^{\infty}\varphi_{\alpha,\nu-\alpha}(n,j)\ (1+\omega)^{-(j+1)} \\
&= M\mathcal{E}_{\alpha,\nu}(-\omega,n+1),
\end{aligned}$$

whence it follows that

$$\parallel S_{\alpha,\nu}(n) \parallel\leq M\mathcal{E}_{\alpha,\nu}(-\omega,n+1).$$

Hence, using (1.19), we can get the first desired conclusion. In order to prove the asymptotic behavior for $0<\nu<1$, we notice that from Theorem 1.2.2, we have

$$\mathcal{E}_{\alpha,\nu}(-\omega,n+1)=\int_0^{\infty}p_n(t)t^{\nu-1}E_{\alpha,\nu}(-\omega t^{\alpha})dt. \tag{3.6}$$

Using the estimate (1.6) in (3.6), we obtain

$$\begin{aligned}
|\mathcal{E}_{\alpha,\nu}(-\omega, n+1)| &\leq \int_0^\infty \frac{e^{-t}}{n!} t^{n+\nu-1} E_{\alpha,\nu}(-\omega t^\alpha) dt \\
&\leq \int_0^\infty \frac{e^{-t}}{n!} t^{n+\nu-1} \frac{C}{1+\omega t^\alpha} dt \\
&\leq \frac{C}{n!\omega} \int_0^\infty \frac{e^{-t}}{n!} t^{n+\nu-\alpha-1} dt \\
&= \frac{C}{\omega} \frac{\Gamma(n+\nu-\alpha)}{n!},
\end{aligned}$$

where in the last equality we used the property (1.2). Now, taking into account that $\lim_{n\to\infty} \dfrac{\Gamma(n+\gamma)}{\Gamma(n)n^\gamma} = 1$ for all $\gamma \in \mathbb{C}$ (see (1.2)), and since $0 < \nu < 1$, we deduce for $\gamma := \nu - \alpha$ that

$$\frac{\Gamma(n+\nu-\alpha)}{n!} = \frac{\Gamma(n+\nu-\alpha)}{\Gamma(n)n^{\nu-\alpha}} \frac{1}{n^{1-(\nu-\alpha)}} \to 0, \quad (n \to \infty).$$

This proves the claim and the proof is finished. □

3.2 The Hilfer fractional difference operator

In this section, we introduce our definition of the Hilfer fractional difference operator $\Delta^{\alpha,\beta}$ of order $0 < \alpha$ and type $0 \leq \beta \leq 1$ as follows.

Definition 3.2.1. *The Hilfer fractional difference $\Delta^{\alpha,\beta}$ of order $\alpha > 0$ and type $0 \leq \beta \leq 1$ of a sequence $f \in s(\mathbb{N}_0; X)$ is defined by*

$$\Delta^{\alpha,\beta} f(n) := \Delta^{-\beta(m-\alpha)} \left(\Delta^m \left(\Delta^{-(m-\alpha)(1-\beta)} f\right)\right)(n), \quad n \in \mathbb{N}_0,$$

where $m - 1 < \alpha \leq m$, $m := \lceil \alpha \rceil$.

Note that as expected

$$\begin{aligned}
\Delta^{m,\beta} f(n) &= \Delta^m f(n), \\
\Delta^{\alpha,0} f(n) &= {}_{RL}\Delta^\alpha f(n), \\
\Delta^{\alpha,1} f(n) &= {}_{C}\Delta^\alpha f(n).
\end{aligned}$$

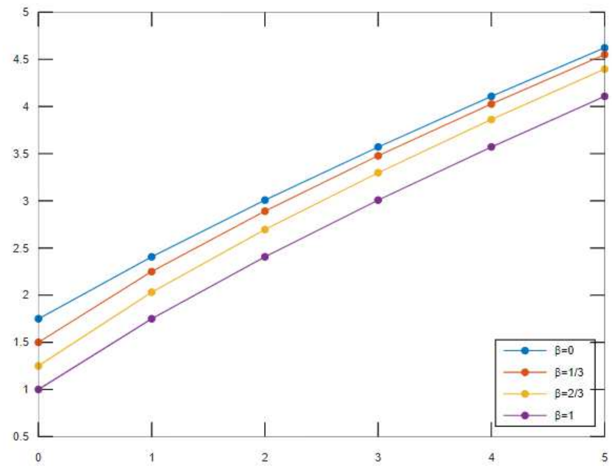

Figure 3.1: $\Delta^{\alpha,\beta} f(n)$ where $\alpha = 1/4$ and $f(n) = n + 1$

.

In other words, the two parameter family of operators $\Delta^{\alpha,\beta}$ of order $\alpha > 0$ and type $0 \leq \beta \leq 1$ allow us to interpolate between the Riemann-Liouville and the Caputo fractional difference operators.

We remark that Definition 3.2.1 can be compared with that recently introduced in [44, Definition 3.1], but first we note that in the definition given by the authors in [44] there is a minor imprecision that we want to clarify. It concerns with the compatibility of the different operators used, because the meaning of the operator Δ used in [44, Definition 3.1] is not precise at all. The definition must read as follows (in case $0 < \mu < 1$, and in the general case is completely analogous),

$$\Delta_a^{\mu,\nu} = \Delta_{a+(1-\nu)(1-\mu)}^{-\nu(1-\mu)} \circ \Delta_{a+(1-\nu)(1-\mu)} \circ \Delta_a^{-(1-\nu)(1-\mu)}, \quad 0 \leq \nu \leq 1, \tag{3.7}$$

where

$$\Delta_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s),$$

with $t \in \mathbb{N}_{a+\alpha}$ and $t^{(\alpha)} := \frac{\Gamma(t+1)}{\Gamma(t-\alpha+1)}$, $\alpha > 0$. For the definition of the operator

$\Delta_{a+(1-\nu)(1-\mu)}$ in (3.7) see

$$\Delta_a^m f(a+n) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} f(a+n+j), \quad n \in \mathbb{N}_0. \tag{3.8}$$

In order to compare both definitions, we define the translation (by $a \in \mathbb{R}$) operator $\tau_a : s(\mathbb{N}_a; X) \to s(\mathbb{N}_a; X)$ by

$$\tau_a g(n) := g(a+n), \; n \in \mathbb{N}_0.$$

Further, we will need the following lemma.

Lemma 3.2.2. *For all $\alpha > 0$ and $b \in \mathbb{R}$, we have $\tau_{b+\alpha} \circ \Delta_b^{-\alpha} = \Delta^{-\alpha} \circ \tau_b$.*

Proof. By definition, for any $f \in s(\mathbb{N}_b, X)$ and all $n \in \mathbb{N}_0$ we have

$$\begin{aligned}
\tau_{b+\alpha} \circ \Delta_b^{-\alpha} f(n) &= \Delta_b^{-\alpha} f(n+b+\alpha) \\
&= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} (\alpha+n-j-1)^{(\alpha-1)} f(b+j) \\
&= \sum_{j=0}^{n} \frac{\Gamma(\alpha+n-j)}{\Gamma(\alpha)\Gamma(n-j+1)} f(b+j) \\
&= \sum_{j=0}^{n} k^\alpha(n-j) f(b+j) = \Delta^{-\alpha} \circ \tau_b f(n).
\end{aligned}$$

□

Remark 3.2.3. *Lemma 3.2.2 shows that the following diagram is commutative:*

$$\begin{array}{ccc}
s(\mathbb{N}_b; X) & \xrightarrow{\Delta_b^{-\alpha}} & s(\mathbb{N}_{b+\alpha}; X) \\
\downarrow \tau_b & & \downarrow \tau_{b+\alpha} \\
s(\mathbb{N}_0; X) & \xrightarrow{\Delta^{-\alpha}} & s(\mathbb{N}_0; X).
\end{array} \tag{3.9}$$

With these preliminaries, we prove the following transference principle that generalizes Theorem 4.1 in [40].

Theorem 3.2.4. *For any $0 < \mu < 1$, $0 \leq \nu \leq 1$ and $a \in \mathbb{R}$ we have*

$$\Delta^{\mu,\nu} = \tau_{a+(1-\mu)(1-\nu)} \circ \Delta_a^{\mu,\nu} \circ \tau_{-a}.$$

Proof. Using (3.8) with $m = 1$, we have

$$\Delta_a^{\mu,\nu} = \Delta_{a+(1-\nu)(1-\mu)}^{-\nu(1-\mu)} \circ \tau_{-(a+(1-\nu)(1-\mu))} \circ \Delta \circ \tau_{a+(1-\nu)(1-\mu)} \circ \Delta_a^{-(1-\nu)(1-\mu)}. \quad (3.10)$$

We now employ (3.9) first with $b = a + (1-\mu)(1-\nu)$, $\alpha = \nu(1-\mu)$ and then with $b = a$, $\alpha = (1-\nu)(1-\mu)$ to obtain

$$\Delta_{a+(1-\nu)(1-\mu)}^{-\nu(1-\mu)} = \tau_{-a-(1-\mu)(1-\nu)} \circ \Delta^{-\nu(1-\mu)} \circ \tau_{a+(1-\mu)(1-\nu)} \quad (3.11)$$

and

$$\Delta_a^{-(1-\nu)(1-\mu)} = \tau_{-a-(1-\nu)(1-\mu)} \circ \Delta^{-(1-\nu)(1-\mu)} \circ \tau_a. \quad (3.12)$$

Replacing (3.11) and (3.12) in (3.10), we obtain

$$\begin{aligned}\Delta_a^{\mu,\nu} &= \tau_{-a-(1-\mu)(1-\nu)} \circ \Delta^{-\nu(1-\mu)} \circ \tau_{a+(1-\mu)(1-\nu)} \circ \tau_{-(a+(1-\nu)(1-\mu))} \circ \Delta \\ &\quad \circ \tau_{a+(1-\nu)(1-\mu)} \circ \tau_{-a-(1-\nu)(1-\mu)} \circ \Delta^{-(1-\nu)(1-\mu)} \circ \tau_a \\ &= \tau_{-a-(1-\mu)(1-\nu)} \circ \Delta^{-\nu(1-\mu)} \circ \Delta \circ \Delta^{-(1-\nu)(1-\mu)} \circ \tau_a = \tau_{-a-(1-\mu)(1-\nu)} \circ \Delta^{\mu,\nu} \circ \tau_a,\end{aligned}$$

proving the theorem. ☐

We obtain the following relation between the Hilfer and Riemann-Liouville fractional difference operators.

Theorem 3.2.5. *Let $n \in \mathbb{N}_0$ and $0 \leq \beta \leq 1$. For each $\alpha > 0$ and $f \in s(\mathbb{N}_0; X)$, we have*

$$\begin{aligned}\Delta^{\alpha,\beta} f(n) &= {}_{RL}\Delta^{\alpha} f(n) \\ &\quad - \sum_{i=0}^{m-1} (-1)^i \binom{m}{i} \sum_{j=0}^{m-1-i} k^{\beta(m-\alpha)}(n+m-j-i) \Delta^{-(m-\alpha)(1-\beta)} f(j),\end{aligned}$$

where $m = \lceil \alpha \rceil$.

Proof. Define $\omega(n) := \Delta^{-(1-\alpha)(1-\beta)} f(n)$. Then, using the definitions, we obtain the following identities

$$\begin{aligned}
\Delta^{\alpha,\beta} f(n) &= \sum_{j=0}^{n} k^{\beta(m-\alpha)}(n-j)\Delta^m \omega(j) \\
&= \sum_{j=0}^{n} k^{\beta(m-\alpha)}(n-j) \sum_{i=0}^{m} (-1)^i \binom{m}{i} \omega(j+m-i) \\
&= \sum_{i=0}^{m} (-1)^i \binom{m}{i} \sum_{j=0}^{n} k^{\beta(m-\alpha)}(n-j)\omega(j+m-i) \\
&= \sum_{i=0}^{m} (-1)^i \binom{m}{i} \sum_{j=m-i}^{n+m-i} k^{\beta(m-\alpha)}(n+m-j-i)\omega(j) \\
&= \sum_{i=0}^{m} (-1)^i \binom{m}{i} \sum_{j=0}^{n+m-i} k^{\beta(m-\alpha)}(n+m-i-j)\omega(j) \\
&\quad - \sum_{i=0}^{m-1} (-1)^i \binom{m}{i} \sum_{j=0}^{m-1-i} k^{\beta(m-\alpha)}(n+m-j-i)\omega(j) \\
&= \Delta^m (\Delta^{-\beta(m-\alpha)}\omega)(n) \\
&\quad - \sum_{i=0}^{m-1} (-1)^i \binom{m}{i} \sum_{j=0}^{m-1-i} k^{\beta(m-\alpha)}(n+m-j-i)\omega(j).
\end{aligned}$$

Here, we have adopted the notation $\sum_{j=0}^{-1} f(j) = 0$. The conclusion follows from the property

$$\Delta^{-\alpha}\Delta^{-\beta} = \Delta^{-(\alpha+\beta)}, \quad \alpha, \beta > 0.$$

□

It is instructive to look at the cases $0 < \alpha < 1$ and $1 < \alpha < 2$ for further developments.

Corollary 3.2.6. *Let* $0 \leq \beta \leq 1$. *For* $0 < \alpha < 1$, *we have*

$$\Delta^{\alpha,\beta} f(n) = {}_{RL}\Delta^{\alpha} f(n) - k^{\beta(1-\alpha)}(n+1)f(0), \qquad n \in \mathbb{N}_0, \tag{3.13}$$

and in case $1 < \alpha < 2$, *we obtain*

$$\Delta^{\alpha,\beta} f(n) = {}_{RL}\Delta^{\alpha} f(n) - k^{\beta(2-\alpha)}(n+1)\left[\Delta^{-(2-\alpha)(1-\beta)} f(1) - 2f(0)\right] + k^{\beta(2-\alpha)}(n+2)f(0),$$

for all $n \in \mathbb{N}_0$.

Concerning to the $\mathcal{Z}$-transform of the Hilfer fractional difference operator, we prove the following property. For it, we recall that the $\mathcal{Z}$-transforms of the Caputo and Riemann-Liouville fraccional forward difference operators are given by:

$$\widetilde{\Delta^{-\alpha} f}(z) = \left(\frac{z}{z-1}\right)^{\alpha} \widetilde{f}(z), \quad |z| > 1, \tag{3.14}$$

$$\widetilde{{}_{RL}\Delta^{\alpha} f}(z) = z\left(\frac{z}{z-1}\right)^{-\alpha} \widetilde{f}(z) - zf(0), \quad |z| > 1, \tag{3.15}$$

where $\alpha \in (0, 1]$, see [67, Propositions 4 and 24].

Proposition 3.2.7. *For* $0 < \alpha < 1$ *and* $0 \le \beta \le 1$, *let* $y(n) := \Delta^{\alpha,\beta} f(n)$ *where* $f \in s(\mathbb{N}_0; X)$. *Then*

$$\widetilde{y}(z) = z^{1-\alpha}(z-1)^{\alpha}\tilde{f}(z) - z^{\beta(1-\alpha)+1}(z-1)^{\alpha-\nu} f(0), \quad |z| > 1,$$

where $\nu = \alpha + \beta(1-\alpha)$.

Proof. According to Definition 3.2.1, we have

$$y(n) = \Delta^{-\beta(1-\alpha)} {}_{RL}\Delta^{\nu} f(n).$$

Let $r(n) := {}_{RL}\Delta^{\nu} f(n)$. From (3.14) we obtain

$$\widetilde{\Delta^{-\beta(1-\alpha)} r}(z) = z^{\beta(1-\alpha)}(z-1)^{\beta(\alpha-1)}\tilde{r}(z), \tag{3.16}$$

and, by (3.15),

$$\widetilde{r}(z) = z^{1-\nu}(z-1)^{\nu}\tilde{f}(z) - zf(0). \tag{3.17}$$

Substituting (3.17) in (3.16), we get the conclusion. $\square$

Remark 3.2.8. *The above proposition coincides with [44, Theorem 3.5] (and also rectifies the mentioned result in [44]), after using the transference principle given by Theorem 3.2.4. We will also need the following lemma that connects the Delta Laplace transform $\mathcal{L}_b$ [41, Definition 2.1] with the usual $\mathcal{Z}$-transform.*

Lemma 3.2.9. *Assume $f : \mathbb{N}_b \to X$. Then*

$$\mathcal{L}_b\{f\}(z-1) = \frac{1}{z}\widetilde{(\tau_b f)}(z),$$

for all $z \in \mathbb{C} \setminus \{0\}$ such that the series defined by the $\mathcal{Z}$-transform converges.

Proof. Using [41, Theorem 2.2], we get

$$\mathcal{L}_b\{f\}(z-1) = \sum_{k=0}^{\infty} \frac{f(b+k)}{z^{k+1}} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{\tau_b f(k)}{z^k} = \frac{1}{z}\widetilde{(\tau_b f)}(z).$$

□

Remark 3.2.10. *By Lemma 3.2.9, Theorem 3.2.4 and Proposition 3.2.7, we have the following identities*

$$\begin{aligned}
\mathcal{L}_{a+(1-\mu)(1-\nu)}\{\Delta_a^{\mu,\nu} f\}(z-1) &= \frac{1}{z}(\tau_{a+(1-\mu)(1-\nu)} \circ \Delta_a^{\mu,\nu} \tilde{f})(z) = \frac{1}{z}(\Delta^{\mu,\nu} \circ \tau_a \tilde{f})(z) \\
&= z^{1-\mu}(z-1)^{\mu}\frac{1}{z}\widetilde{(\tau_a f)}(z) - z^{\nu(1-\mu)}(z-1)^{-\nu(1-\mu)}(\tau_a f)(0) \\
&= z^{1-\mu}(z-1)^{\mu}\mathcal{L}_a\{f\}(z-1) - z^{\nu(1-\mu)}(z-1)^{-\nu(1-\mu)} f(a).
\end{aligned}$$

Replacing $s = z-1$, we improve [44, Theorem 3.5] as follows:

$$\mathcal{L}_{a+(1-\mu)(1-\nu)}\{\Delta_a^{\mu,\nu} f\}(s) = (s+1)^{1-\mu} s^{\mu}\mathcal{L}_a\{f\}(s) - (s+1)^{\nu(1-\mu)} s^{-\nu(1-\mu)} f(a).$$

Now, employing the Poisson transform, we can establish an important relation between the Hilfer fractional difference operator and the Hilfer fractional continuous operator. This result extends [56, Theorem 3.5].

Theorem 3.2.11. *Let $u : [0,\infty) \to X$ be an absolutely integrable and bounded function. Then*

$$\mathcal{P}({}_HD^{\alpha,\beta}u)(n+1) = \Delta^{\alpha,\beta}\mathcal{P}(u)(n), \quad n \in \mathbb{N}_0, \tag{3.18}$$

that is

$$\int_0^\infty p_{n+1}(t)\,{}_HD_t^{\alpha,\beta}u(t)\;dt = \Delta^{\alpha,\beta}u(n), \qquad n \in \mathbb{N}_0,$$

where $u(n) := \int_0^\infty p_n(t)u(t)\;dt$.

Proof. Taking the definition of continuous Hilfer derivative (see (1.3)), multiplying by $p_n(t)$ and then integrating over $\mathbb{R}_+$, we obtain

$$\int_0^\infty p_{n+1}(t)\,{}_HD_t^{\alpha}u(t)\;dt = \int_0^\infty p_{n+1}(t)(g_{\beta(1-\alpha)} * {}_{RL}D_t^{\nu}u)(t)\;dt,$$

where $\nu = \alpha + \beta(1-\alpha)$. On the other hand, by [56, Theorem 3.4], we have

$$\int_0^\infty p_{n+1}(t)(g_{\beta(1-\alpha)} * {}_{RL}D_t^{\nu}u)(t)\;dt = \sum_{j=0}^{n+1} a(n+1-j)S(j),$$

where

$$a(n) := \int_0^\infty p_n(t)g_{\beta(1-\alpha)}(t)dt \quad \textit{and} \quad S(n) := \int_0^\infty p_n(t)_{RL}D_t^{\nu}u(t)\;dt.$$

Now, by definition of k^γ (see (1.16)), we get

$$a(n) = \frac{1}{n!\Gamma(\beta(1-\alpha))}\int_0^\infty e^{-t}t^{n+\beta(1-\alpha)-1}\;dt = \frac{\Gamma(n+\beta(1-\alpha))}{\Gamma(\beta(1-\alpha))n!} = k^{\beta(1-\alpha)}(n).$$

Therefore, [56, Theorem 3.5] gives

$$\begin{aligned}\int_0^\infty p_{n+1}(t)\,{}_HD_t^{\alpha}u(t)\;dt &= \sum_{j=0}^{n+1} k^{\beta(1-\alpha)}(n+1-j)S(j)\\ &= \Delta^{-\beta(1-\alpha)}S(n+1)\\ &= \Delta^{-\beta(1-\alpha)}{}_{RL}\Delta^{\nu}u(n).\end{aligned}$$

Hence the conclusion follows. ∎

We finish this section with the following result that shows the connection between resolvent sequences and solutions of the abstract Cauchy problem in discrete-time that involves the Hilfer fractional difference operator previously.

Theorem 3.2.12. *Let* $0 < \alpha < 1$, $0 \leq \beta \leq 1$ *and* $\nu = \alpha + \beta(1-\alpha)$. *Suppose that* A *is the generator of an* (α,ν)*-resolvent sequence* $\{S_{\alpha,\nu}(n)\}_{n\in\mathbb{N}_0}$ *and* $1 \in \rho(A)$. *Then the fractional difference equation*

$$\begin{aligned} \Delta^{\alpha,\beta}u(n) &= A\left[u(n+1) - k^{\beta(1-\alpha)}(n+1)u_0\right], \quad n \in \mathbb{N}_0, \\ u(0) &= u_0 \in D(A), \end{aligned} \tag{3.19}$$

admits the solution

$$u(n) = S_{\alpha,\nu}(n)C^{-1}u_0, \quad n \in \mathbb{N}_0,$$

where $C = (I-A)^{-1}$.

Proof. Let $u(n) := S_{\alpha,\nu}(n)C^{-1}u_0$ for all $n \in \mathbb{N}_0$. By Definition 3.1.4, we have $u(n) \in D(A)$ and

$$\begin{aligned} (k^{1-\alpha} * S_{\alpha,\nu})(n)x &= k^{1+\beta(1-\alpha)}(n)x + A(k^1 * S_{\alpha,\nu})(n)x \\ &= k^{1+\beta(1-\alpha)}(n)x + A\sum_{j=0}^{n} S_{\alpha,\nu}(j)x, \quad n \in \mathbb{N}_0, \end{aligned}$$

holds. Applying the operator Δ to both sides of the last identity and

$${}_{RL}\Delta^{\sigma}k^{\tau}(n) = k^{\tau}(n+1), \quad \sigma \in (0,1),\ \sigma < \tau,$$

we get

$${}_{RL}\Delta S_{\alpha,\nu}(n)x = k^{\beta(1-\alpha)}(n+1)x + AS_{\alpha,\nu}(n+1)x, \quad n \in \mathbb{N}_0.$$

Then, by Corollary 3.2.6 and the identity $C - I = AC$, we obtain

$$\begin{aligned} \Delta^{\alpha,\beta}S_{\alpha,\nu}(n) &= k^{\beta(1-\alpha)}(n+1)x + AS_{\alpha,\nu}(n+1)x - k^{\beta(1-\alpha)}(n+1)S_{\alpha,\nu}(0)x \\ &= AS_{\alpha,\nu}(n+1)x - k^{\beta(1-\alpha)}(n+1)(C-I)x \\ &= A\left[S_{\alpha,\nu}(n+1)x - k^{\beta(1-\alpha)}(n+1)Cx\right], \quad n \in \mathbb{N}_0. \end{aligned}$$

Hence, u solves (3.19), which proves the Theorem. $\square$

Remark 3.2.13. *Under the same hypotheses of the previous theorem, note that* $u(n) := S_{\alpha,\alpha}(n)C^{-1}u_0$ *solves*

$$\begin{aligned} {}_{RL}\Delta^{\alpha}u(n) &= Au(n+1), \quad n \in \mathbb{N}_0, \\ u(0) &= u_0 \in D(A), \end{aligned}$$

and $u(n) := S_{\alpha,1}(n)C^{-1}u_0$ *solves*

$$\begin{aligned} {}_{C}\Delta^{\alpha}u(n) &= A\left[u(n+1) - k^{1-\alpha}(n+1)u_0\right], \quad n \in \mathbb{N}_0, \\ u(0) &= u_0 \in D(A). \end{aligned}$$

The following corollary is an interesting but direct consequence of the theory developed until now. It takes in consideration the subordination formula stated previously.

Corollary 3.2.14. *Let* $0 < \alpha < 1$ *and* $0 < \beta < 1$ *be given. Assume that* A *generates a discrete* C*-semigroup* $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0}$ *such that* $\|\mathcal{T}(n)\| \le M(1+\omega)^{-n}$ *for some* $M, \omega > 0$. *Then the fractional difference equation*

$$\begin{aligned} \Delta^{\alpha,\beta}u(n) &= A\left[u(n+1) - k^{\beta(1-\alpha)}(n+1)u_0\right], \quad n \in \mathbb{N}_0, \\ u(0) &= u_0 \in D(A), \end{aligned} \tag{3.20}$$

admits the solution

$$u(n) = \sum_{j=0}^{\infty} \varphi_{\alpha,\beta(1-\alpha)}(n,j)\mathcal{T}(j)C^{-1}u_0, \quad n \in \mathbb{N}_0.$$

Moreover, $u(n) \to 0$ *as* $n \to \infty$.

Proof. Since A generates a discrete C-semigroup we have $1 \in \rho(A)$. Let $\nu := \alpha + \beta(1-\alpha)$. By Theorem 3.1.5 (subordination) we have that A generates a discrete (α,ν)-resolvent family $\{S_{\alpha,\nu}(n)\}_{n\in\mathbb{N}_0}$ given by

$$S_{\alpha,\nu}(n)x = \sum_{j=0}^{\infty} \varphi_{\alpha,\nu-\alpha}(n,j)\mathcal{T}(j)x, \quad n \in \mathbb{N}_0. \tag{3.21}$$

From Theorem 3.2.12 the unique solution is given by

$$u(n) = S_{\alpha,\nu}(n)C^{-1}u_0, \quad n \in \mathbb{N}_0,$$

where $C = (I - A)^{-1}$. Since $\nu \geq \alpha$, Proposition 3.1.10 implies that $\|S_{\alpha,\nu}(n)\| \leq M\mathcal{E}_{\alpha,\nu}(-\omega, n)$. Note that for $0 < \beta < 1$ we have $\alpha(1-\beta) < 1-\beta$ and hence $0 < \nu < 1$. It follows from Proposition 3.1.10 that $u(n) \to 0$ as $n \to \infty$, finishing the proof. $\square$

Remark 3.2.15. *Let us consider the discrete fractional diffusion problem*

$$\begin{aligned} {}_{RL}\Delta^{\alpha} v(n,x) &= \mathbf{\Delta}_x v(n+1,x), \;\; n \in \mathbb{N}_0, \;\; x \in \mathbb{R}^d, \\ u(0,x) &= f(x), \end{aligned} \tag{3.22}$$

where $0 < \alpha < 1$. Then, by Theorem 3.1.5 we have that $\mathbf{\Delta}_x$ generates an (α,α)-resolvent sequence $\{S_{\alpha,\alpha}(n)\}_{n\in\mathbb{N}_0}$ with $C^{-1} = I - \mathbf{\Delta}_x$ and moreover, by Remark 3.2.13, the solution of (3.22) is given by

$$v(n,x) = \sum_{j=0}^{\infty} \varphi_{\alpha,0}(n,j)(\mathcal{G}_{j+1} \circledast \phi)(x),$$

where $\phi(x) := C^{-1}f(x)$ and $\mathcal{G}_n$ is (2.7).

4. Existence and uniqueness of (N, λ)-periodic solutions to a class of Volterra difference equations

In this chapter, we introduce the class of (N, λ)-periodic vector-valued sequences and show several notable properties of this new class. This class includes periodic, anti-periodic, Bloch and unbounded sequences. Furthermore, we show the existence and uniqueness of (N, λ)-periodic solutions to the following class of Volterra difference equations with infinite delay

$$u(n+1) = \sigma \sum_{j=-\infty}^{n} a(n-j)u(j) + f(n, u(n)), \quad n \in \mathbb{Z}, \quad \sigma \in \mathbb{C},$$

where the kernel a and the nonlinear term f satisfy suitable conditions.

4.1 (N, λ)-periodic discrete functions

In this section, we introduce the concept of (N, λ)-periodic discrete vector-valued function and show some remarkable properties of this class of vector-valued sequences.

Definition 4.1.1. *A vector-valued function $f : \mathbb{Z} \to X$ is called (N, λ)-periodic discrete function (or (N, λ)-periodic sequence) if there exist $N \in \mathbb{Z}_+$ and $\lambda \in \mathbb{C} \setminus \{0\}$ such that $f(n + N) = \lambda f(n)$ for all $n \in \mathbb{Z}$. N is called the λ-period of*

f. The collection of those sequences with the same λ-period N will be denoted by $\mathbb{P}_{N\lambda}(\mathbb{Z},X)$.

In case $\lambda = 1$ we denote simply by $\mathbb{P}_N(\mathbb{Z},X)$ the set of all N-periodic sequences.

The following property gives a useful characterization of (N,λ)-periodic discrete functions.

Proposition 4.1.2. *A function f is (N,λ)-periodic discrete function, if and only if there exists $u \in \mathbb{P}_N(\mathbb{Z},X)$ such that*

$$f(n) = \lambda^\wedge(n)u(n), \qquad \text{for all } n \in \mathbb{Z}, \tag{4.1}$$

where $\lambda^\wedge(n) := \lambda^{n/N}$.

Proof. First, we assume that $f \in \mathbb{P}_{N\lambda}(\mathbb{Z},X)$ and define $u(n) := \lambda^\wedge(-n)f(n)$. Then,

$$u(n+N) = \lambda^\wedge(-(n+N))f(n+N) = \lambda^\wedge(-n)f(n) = u(n).$$

Hence $u \in \mathbb{P}_N(\mathbb{Z},X)$ and $f(n) = \lambda^\wedge(n)u(n)$. Conversely, we suppose $f(n) = \lambda^\wedge(n)u(n)$. Then

$$f(n+N) = \lambda^\wedge(n+N)u(n+N) = \lambda \cdot \lambda^\wedge(n)u(n) = \lambda f(n).$$

$\square$

Example 4.1.3. *The function $f(n) = \cos(\pi n/6)$ is an $(6,-1)$-periodic discrete function. It follows from Proposition 4.1.2 that f has decomposition $f(n) = \lambda^\wedge(n)u(n)$ where*

$$\lambda^\wedge(n) = (-1)^{n/6} = \cos(n\pi/6) + i\sin(n\pi/6),$$

and

$$u(n) = (-1)^{-n/6}f(n) = \cos(n\pi/6)[\cos(n\pi/6) - i\sin(n\pi/6)].$$

Example 4.1.4. *Let $\mathcal{A}$ be a $k \times k$ matrix. Assume that there exists $N \in \mathbb{Z}_+$ (sufficiently large) such that $\mathcal{A}(n+N) = \mathcal{A}(n)$ for all $n \in \mathbb{Z}_+$. Let $\mathcal{K}$ be the $k \times k$ matrix defined as follows:*

$$\mathcal{K} := \prod_{i=0}^{N-1} \mathcal{A}(i), \quad n \in \mathbb{Z}_+,$$

where $\prod_{i=0}^{N-1} \mathcal{A}(i) := \mathcal{A}(N-1)\mathcal{A}(N-2)\cdots\mathcal{A}(0)$. Furthermore, let $\lambda_0 \in \mathbb{C} \setminus \{0\}$ be any eigenvalue of $\mathcal{K}$ with corresponding eigenvector X_0. It can be proved that the solution of the system

$$\begin{aligned} U(n+1) &= \mathcal{A}(n)U(n), \qquad \text{for } n \in \mathbb{Z}_+ \\ U(0) &= X_0, \end{aligned} \tag{4.2}$$

is given by

$$U(n) = \prod_{i=0}^{n-1} \mathcal{A}(i)X_0. \tag{4.3}$$

Moreover,

$$\begin{aligned} U(n+N) &= \prod_{i=0}^{n+N-1} \mathcal{A}(i)X_0 = \prod_{i=N}^{n+N-1} \mathcal{A}(i) \prod_{i=0}^{N-1} \mathcal{A}(i)X_0 \\ &= \prod_{i=N}^{n+N-1} \mathcal{A}(i)\mathcal{K}\lambda_0 = \prod_{i=0}^{n-1} \mathcal{A}(i)\lambda_0 X_0 \\ &= \lambda_0 \prod_{i=0}^{n-1} \mathcal{A}(i)X_0 = \lambda_0 U(n). \end{aligned}$$

Hence the system (4.2) has a (N, λ_0)-periodic solution given by (4.3). Moreover, $U(n) = \lambda_0^{\wedge}(n)P(n)$ where $P(n) := \lambda_0^{\wedge}(-n)\prod_{i=0}^{n-1} \mathcal{A}(i)X_0$ is a periodic sequence of period N. As a particular example, if

$$\mathcal{A}(n) = \begin{pmatrix} 0 & \frac{3+(-1)^n}{2} \\ \frac{3-(-1)^n}{2} & 0 \end{pmatrix},$$

we have that $N = 2$ and the eigenvalues of $\mathcal{K} := \mathcal{A}(1)\mathcal{A}(0)$ are $\lambda_1 = 1$ and $\lambda_2 = 4$ with the corresponding eigenvectors

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

respectively. If $X(0) = X_1$, then the system (4.2) has a $(2,1)$-periodic solution and, if $X(0) = X_2$, it has a $(2,4)$-periodic solution.

Next, we present some algebraic properties of the (N,λ)-periodic discrete functions.

Theorem 4.1.5. *Let f and g be (N,λ)-periodic discrete functions, $c \in \mathbb{C}$ and $l \in \mathbb{Z}$. Then the following assertions are valid:*

(i) *$w := f + g$ is a (N,λ)-periodic discrete function.*

(ii) *$p := cf$ is a (N,λ)-periodic discrete function.*

(iii) *For each fixed l in $\mathbb{Z}$ the function $f_l : \mathbb{Z} \to X$ defined by $f_l(n) := f(n+l)$ is a (N,λ) periodic discrete function.*

Proof. The proof is immediate. Indeed, for all $n \in \mathbb{Z}$ we have that

(i) $w(n+N) = (f+g)(n+N) = \lambda\, w(n)$.

(ii) $p(n+N) = (cf)(n+N) = \lambda\, p(n)$.

(iii) $f_l(n+N) = f(n+N+l) = f(n_0+N) = \lambda\, f_l(n)$.

□

Theorem 4.1.6. *Let $f \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$, then $\Delta f \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$.*

Proof. Since $\Delta f(n) = f(n+1) - f(n)$, then by (i) and (iii) of Theorem 4.1.5, we have that Δf is a (N,λ)-periodic discrete function. □

In order to give a Banach structure to the vector space $\mathbb{P}_{N\lambda}(\mathbb{Z}, X)$, we need to define a suitable norm. We recall that the space of N-periodic discrete functions equipped with the norm

$$\|u\|_N := \max_{n\in[0,N]\cap\mathbb{Z}} \|u(n)\|_X \tag{4.4}$$

is a Banach space.

Proposition 4.1.7. *$\mathbb{P}_{N\lambda}(\mathbb{Z},X)$ is a Banach space with the norm*

$$\| f \|_{N\lambda} := \max_{n\in[0,N]\cap\mathbb{Z}} \| \lambda^\wedge(-n) f(n) \|_X \,. \tag{4.5}$$

Proof. The proof follows from Proposition 4.1.2 and the fact that $\mathbb{P}_N(\mathbb{Z},X)$ is a Banach space with the norm (4.4). ◻

Next, we present a convolution theorem. This result is a useful tool in order to study the existence and uniqueness of (N,λ)-periodic discrete solutions of abstract Volterra difference equations.

Theorem 4.1.8. *Let $f \in \mathbb{P}_{N\lambda}(\mathbb{Z},X)$ and assume that $b:\mathbb{N}_0 \to \mathbb{C}$ is such that the sequence $b^{\smile}(n) := \lambda^\wedge(-n)b(n)$ is summable. Then $b \star f$ defined by*

$$(b\star f)(n) = \sum_{j=-\infty}^{n} b(n-j)f(j), \quad n\in\mathbb{Z},$$

is well defined in the norm $\|\cdot\|_{N\lambda}$ and belongs to $\mathbb{P}_{N\lambda}(\mathbb{Z},X)$.

Proof. Let $p(n) := (b\star f)(n)$, $n\in\mathbb{Z}$. First, note that p is well defined in the norm $\|\cdot\|_{N\lambda}$. Indeed,

$$\begin{aligned}
\| \lambda^\wedge(-n)p(n) \|_X &\le \sum_{j=-\infty}^{n} |\lambda^\wedge(-(n-j))b(n-j)|\, \| \lambda^\wedge(-j)f(j) \|_X \\
&\le \| f \|_{N\lambda} \sum_{j=-\infty}^{n} |\lambda^\wedge(-(n-j))b(n-j)| \\
&= \| f \|_{N\lambda} \sum_{j=0}^{\infty} |b^{\smile}(j)|.
\end{aligned}$$

Therefore, $\|p\|_{N\lambda} \leq \|f\|_{N\lambda} \|b^{\smile}\|_{\ell_1}$. Next, we prove that p is (N, λ)-periodic discrete. In fact,

$$\begin{aligned} p(n+N) &= \sum_{j=-\infty}^{n+N} b(n+N-j)f(j) = \sum_{j=-\infty}^{n+N} b(n-(j-N))f(j) \\ &= \sum_{r=-\infty}^{n} b(n-r)f(r+N) = \lambda \sum_{r=-\infty}^{n} b(n-r)f(r) = \lambda\, p(n). \end{aligned}$$

Hence $p \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$. $\square$

In order to prove the next composition result, we need the following useful lemma.

Lemma 4.1.9. *For every $(m, x) \in \mathbb{Z} \times X$, there exists $\phi \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$ such that*

$$\phi(m) = x.$$

Proof. It is enough to consider $\phi(n) := \lambda\hat{}(n-m)x$. $\square$

Let $g : \mathbb{Z} \times X \to X$ and $\phi \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$. We recall that the operator $\mathcal{N}(\phi)(\cdot) := g(\cdot, \phi(\cdot))$ is called the Nemytskii discrete composition operator. We study the invariance of $\mathcal{N}$ on $\mathbb{P}_{N\lambda}(\mathbb{Z}, X)$.

Theorem 4.1.10. *Let $g : \mathbb{Z} \times X \to X$. Then the following assertions are equivalent:*

(i) for every $\phi \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$ we have that $\mathcal{N}(\phi)$ is (N, λ)-periodic discrete.

(ii) g is N-periodic in the first variable and homogeneous in the second variable, that is $g(n+N, \lambda x) = \lambda g(n, x)$ for all $(n, x) \in \mathbb{Z} \times X$.

Proof. Assume (ii). Then for $\phi \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$ and all $n \in \mathbb{Z}$, we have

$$\mathcal{N}(\phi)(n+N) = g(n+N, \phi(n+N)) = g(n+N, \lambda\phi(n)) = \lambda\, \mathcal{N}(\phi)(n).$$

Thus, we conclude that $\mathcal{N}(\phi) \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$. Suppose (i) and let $(n, x) \in \mathbb{Z} \times X$ be arbitrary. By Lemma 4.1.9, there exists $\phi \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$ such that $\phi(n) = x$.

Therefore, for such ϕ we have

$$\begin{aligned}\lambda g(n,x) = \lambda\ g(n,\phi(n)) = &\ \lambda\,\mathcal{N}(\phi)(n)\\ = &\ \mathcal{N}(\phi)(n+N) = \ g(n+N,\phi(n+N))\\ = g(n+N,\lambda\phi(n)) = &\ g(n+N,\lambda x),\end{aligned}$$

which gives the claim. □

4.2 Abstract Volterra difference equations

In this section, we establish the existence of (N,λ)-periodic discrete solutions for the following class of linear Volterra difference equations defined on a Banach space X (see [31])

$$u(n+1) = \sigma \sum_{j=-\infty}^{n} a(n-j)u(j) + f(n), \quad n \in \mathbb{Z}, \tag{4.6}$$

where σ is a given complex number, a is summable and $f \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$ for N, λ fixed. Let $\mathcal{S}(\sigma, k)$ be the solution of the difference equation

$$\mathcal{S}(\sigma, n+1) = \sigma \sum_{j=0}^{n} a(n-j)\mathcal{S}(\sigma, j), \qquad n \in \mathbb{N}_0, \tag{4.7}$$

$$\mathcal{S}(\sigma, 0) = 1,$$

and define the set

$$\Omega^N_{\lambda\mathcal{S}} := \left\{\sigma \in \mathbb{C} : \sum_{j=0}^{\infty} |\mathcal{S}^{\smallfrown}(\sigma, j)| < \infty\right\},$$

where $\mathcal{S}^{\smallfrown}(\sigma, j) = \lambda^\wedge(-j)\mathcal{S}(\sigma, j)$. Note that $0 \in \Omega^N_{\lambda\mathcal{S}}$.

Theorem 4.2.1. *Let $a : \mathbb{N}_0 \to \mathbb{C}$ and $f \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$ be given. Suppose that a is summable and $\sigma \in \Omega^N_{\lambda\mathcal{S}}$. Then there is a (N,λ)-periodic discrete solution of (4.6) given by*

$$u(n+1) = \sum_{j=-\infty}^{n} \mathcal{S}(\sigma, n-j)f(j). \tag{4.8}$$

Proof. Since $f \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$ and $\sigma \in \Omega^N_{\lambda\mathcal{S}}$, applying Theorem 4.1.8 we obtain that (4.8) is a well defined (N,λ)-periodic discrete function. Moreover, since a is summable, following the same lines that in [31, Theorem 3.1], we find that u satisfies (4.6). Indeed,

$$
\begin{aligned}
\sigma \sum_{j=-\infty}^{n} a(n-j)u(j) + f(n) \\
&= \sigma \sum_{j=-\infty}^{n} a(n-j)\left(\sum_{i=-\infty}^{j-1} \mathcal{S}(\sigma, j-1-i)f(i)\right) + f(n) \\
&= \sigma \sum_{j=-\infty}^{n-1} \sum_{i=-\infty}^{j} a(n-1-j)\mathcal{S}(\sigma, j-i)f(i) + f(n) \\
&= \sigma \sum_{i=-\infty}^{n-1} \sum_{j=i}^{n-1} a(n-1-j)\mathcal{S}(\sigma, j-i)f(i) + f(n) \\
&= \sigma \sum_{i=-\infty}^{n-1} \left(\sum_{j=0}^{n-1-i} a(n-1-i-j)\mathcal{S}(\sigma, j)\right) f(i) + f(n) \\
&= \sum_{i=-\infty}^{n-1} \mathcal{S}(\sigma, n-i)f(i) + \mathcal{S}(\sigma, 0)f(n) \\
&= \sum_{i=-\infty}^{n} \mathcal{S}(\sigma, n-i)f(i) = u(n+1).
\end{aligned}
$$

□

Remark 4.2.2. *Uniqueness of solutions to the linear case follows directly from [12, Remark 2.4].*

Now, we consider the problem of existence and uniqueness of (N,λ)-periodic discrete solutions for the class of semilinear Volterra difference equations on a Banach space X given by

$$
u(n+1) = \sigma \sum_{j=-\infty}^{n} a(n-j)u(j) + f(n, u(n)), \quad n \in \mathbb{Z}, \tag{4.9}
$$

where $\sigma \in \mathbb{C}$ and f satisfies suitable conditions. Here, we assume that

$$
a^{\frown}(k) := \lambda^{\wedge}(-k)a(k), \;\; k \in \mathbb{N}_0
$$

is such that

$$\|a^{\smallsmile}\|_{\ell_1} < \infty.$$

For example, if a is summable and $|\lambda| \geq 1$ then $a^{\smallsmile} \in \ell_1(\mathbb{N}_0)$.

Theorem 4.2.3. *Let $f : \mathbb{Z} \times X \to X$ be given. Assume the following conditions:*

(i) There exists $(N,\lambda) \in \mathbb{Z} \times (\mathbb{C} \setminus \{0\})$ such that

$$f(n+N, \lambda x) = \lambda f(n,x)$$

for all $(n,x) \in \mathbb{Z} \times X$.

(ii) There exists a constant $L > 0$ such that

$$\| f(n,x) - f(n,y) \|_X \leq L \| x - y \|_X$$

for all $x, y \in X$ and $n \in \mathbb{Z}$.

(iii) $\sigma \in \Omega^N_{\lambda\mathcal{S}}$.

(iv) $L \sum_{k=0}^{\infty} |\mathcal{S}^{\smallsmile}(\sigma,k)| < 1$.

Then equation (4.9) has a unique solution in $\mathbb{P}_{N\lambda}(\mathbb{Z},X)$ satisfying

$$u(n+1) = \sum_{j=-\infty}^{n} \mathcal{S}(\sigma, n-j) f(j, u(j)).$$

Proof. We define the operator $G : \mathbb{P}_{N\lambda}(\mathbb{Z},X) \to \mathbb{P}_{N\lambda}(\mathbb{Z},X)$ by

$$G(u)(n) := \sum_{j=-\infty}^{n} \mathcal{S}(\sigma, n-j) f(j, u(j)).$$

By hypothesis (i), Theorem 4.1.10 and Theorem 4.1.8 we have that $G(u)$ is a (N,λ)-periodic discrete function and therefore G is well defined. Now, for $u, v \in \mathbb{P}_{N\lambda}(\mathbb{Z},X)$ we get by hypothesis (ii)

$$
\begin{aligned}
&\| \lambda^\wedge(-n) \sum_{j=-\infty}^{n} \mathcal{S}(\sigma, n-j) \left[f(j, u(j) - f(j, v(j)) \right] \|_X \\
&\le \sum_{j=-\infty}^{n} |\lambda^\wedge(-(n-j))\mathcal{S}(\sigma, n-j)| \; \| \lambda^\wedge(-j) \left[f(j, u(j) - f(j, v(j)) \right] \|_X \\
&\le \sum_{j=-\infty}^{n} |\lambda^\wedge(-(n-j))\mathcal{S}(\sigma, n-j)||\lambda|^{-j/N} \; \| \left[f(j, u(j) - f(j, v(j)) \right] \|_X \\
&\le L \sum_{j=-\infty}^{n} |\lambda^\wedge(-(n-j))\mathcal{S}(\sigma, n-j)||\lambda|^{-j/N} \; \| \left[u(j) - v(j) \right] \|_X \\
&= L \sum_{j=-\infty}^{n} |\lambda^\wedge(-(n-j))\mathcal{S}(\sigma, n-j)||\lambda|^{-j/N} \; \| \lambda^\wedge(j)\lambda^\wedge(-j) \left[u(j) - v(j) \right] \|_X \\
&= L \sum_{j=-\infty}^{n} |\lambda^\wedge(-(n-j))\mathcal{S}(\sigma, n-j)| \; \| \lambda^\wedge(-j) \left[u(j) - v(j) \right] \|_X \\
&\le \| u - v \|_{N\lambda} \, L \sum_{k=0}^{\infty} |\mathcal{S}^\smile(\sigma, k)|.
\end{aligned}
$$

By (iii) and (iv), we obtain

$$
\begin{aligned}
\| G(u) - G(v) \|_{N\lambda} &= \max_{n \in [0,N] \cap \mathbb{Z}} \| \lambda^\wedge(-n) \sum_{j=-\infty}^{n} \mathcal{S}(\sigma, n-j) \left[f(j, u(j) - f(j, v(j)) \right] \|_X \\
&\le \| u - v \|_{N\lambda} \, L \sum_{k=0}^{\infty} |\mathcal{S}^\smile(\sigma, k)|.
\end{aligned}
$$

It follows that G is a contraction. Then there exists a unique function $u \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$ such that $Gu = u$. Hence u is the unique solution of equation (4.9). $\square$

Example 4.2.4. *We consider the following difference equation in the Banach space $X = \mathbb{R}$,*

$$
u(n+1) = \sigma \sum_{j=-\infty}^{n} p^{n-j} u(j) + \nu g(n) \;\; \cos(h(n)u(n)), \quad n \in \mathbb{Z}, \tag{4.10}
$$

where $g \in \mathbb{P}_{N\lambda}(\mathbb{Z}, \mathbb{R})$, $h \in \mathbb{P}_{N\frac{1}{\lambda}}(\mathbb{Z}, \mathbb{R})$, $p \in \mathbb{C}$ is such that $|p| < 1$ and

$$
\sigma \in \mathcal{M} := \{ z \in \mathbb{C} : \;\; |z + p| < |\lambda|^{1/N} \}.
$$

Let $\varphi(n) := g(n)h(n)$. Note that φ is a periodic function with N period. Then, there exists a constant τ such that $\tau := \max_{n\in[0,N]\cap\mathbb{Z}} |\varphi(n)|$. We claim that if

$$|\nu| < \frac{|\lambda|^{1/N} - |\sigma + p|}{(|\lambda|^{1/N} - |\sigma + p| + |\sigma|)\ |\tau|}, \tag{4.11}$$

then (4.10) has a unique (N,λ)-periodic discrete solution. In order to show this, first, let us determine the solution $\mathcal{S}(\sigma, n)$ of the problem

$$\begin{aligned} &\mathcal{S}(\sigma, n+1) = \sigma \sum_{j=0}^{n} p^{k-j}\mathcal{S}(\sigma, j), \qquad n \in \mathbb{N}_0, \\ &\mathcal{S}(\sigma, 0) = 1, \end{aligned}$$

using the Z-transform. Indeed, we have $z\tilde{\mathcal{S}}(z) - z\mathcal{S}(\sigma, 0) = \sigma\ \tilde{p}(z)\tilde{\mathcal{S}}(z)$ or, equivalently, $z\tilde{\mathcal{S}}(z) - z = \sigma\left(\frac{z}{z-p}\right)\tilde{\mathcal{S}}(z)$. Then,

$$\tilde{\mathcal{S}}(z) = \frac{z}{z - \sigma\left(\frac{z}{z-p}\right)} = \frac{z-p}{z-p-\sigma}.$$

Hence,

$$\mathcal{S}(\sigma, n) = (\sigma + p)^n - p(p+\sigma)^{n-1} = \sigma(\sigma + p)^{n-1}, \qquad n \geq 1. \tag{4.12}$$

It follows that $\sigma \in \mathcal{M} \subset \Omega^N_{\lambda\mathcal{S}}$, which proves condition (iii) of Theorem 5.2.2.

On the other hand, note that $f(n,x) := \nu g(n)\cos(h(n)x)$ satisfies the hypotheses (i) and (ii) of Theorem 5.2.2:

(i)

$$\begin{aligned} f(n+N, \lambda x) &= \nu g(n+N)\cos(h(n+N)\lambda x) = \nu\lambda g(n)\cos\left(\frac{1}{\lambda}h(n)\lambda x\right) \\ &= \lambda\nu g(n)\cos(h(n)x) = \lambda f(n,x). \end{aligned}$$

(ii)

$$|f(n,x) - f(n,y)| \leq |\nu g(n)h(n)||x-y| \leq |\nu\tau||x-y| := L|x-y|.$$

Next, we show part (iv) of Theorem 5.2.2. Indeed, using (4.11) and (4.12) we have that

$$
\begin{aligned}
L\sum_{j=0}^{\infty}|\mathcal{S}^{\smile}(\sigma,j)| &= |\nu\tau|\left(1+\sum_{j=1}^{\infty}|\lambda^{\wedge}(-j)\sigma(\sigma+p)^{n-1}|\right)\\
&= |\nu\tau|\left(1+\frac{|\sigma|}{|\lambda|^{1/N}}\sum_{j=1}^{\infty}\left(\frac{|\sigma+p|}{|\lambda|^{1/N}}\right)^{n-1}\right)\\
&= |\nu\tau|\left(1+\frac{|\sigma|}{|\lambda|^{1/N}-|\sigma+p|}\right)\\
&= |\nu\tau|\frac{|\lambda|^{1/N}-|\sigma+p|+|\sigma|}{|\lambda|^{1/N}-|\sigma+p|}<1.
\end{aligned}
$$

Thus, we have checked all the hypotheses of Theorem 5.2.2. Hence there exists a unique (N,λ)-periodic discrete solution u of (4.10) satisfying

$$u(n+1)=\nu\sum_{j=-\infty}^{n}\mathcal{S}(\sigma,n-j)g(j)\cos(h(j)u(j)).$$

Remark 4.2.5. *As a particular case of the previous example, we can consider the functions $h(n):=(1/2)^{n/8}\sin(n\pi/4)$ and $g(n):=(2)^{n/8}\cos(n\pi/4)$.*

5. Existence and uniqueness of (N,λ)-periodic solutions for abstract fractional difference equation

In this chapter, we establish sufficient conditions for the existence and uniqueness of (N,λ)-periodic solutions for the nonlinear fractional equation

$$\Delta_W^{\alpha} u(n) = Au(n+1) + f(n,u(n)), \quad n \in \mathbb{Z},$$

where A is a closed linear operator with domain $D(A)$ defined on a complex Banach space X equipped with the norm $\|\cdot\|_X$, $0<\alpha\le 1$, Δ_W^{α} denotes the fractional difference operator in the sense of Weyl-like and f satisfies appropriate conditions.

5.1 Linear fractional difference equations

Given $\alpha>0$, we define the vector subspace

$$\Theta_{\alpha}^{1}(\mathbb{Z},X) := \left\{ f \in s(\mathbb{Z},X) \,:\, \sum_{n=-\infty}^{\infty} \|n^{\alpha-1} f(n)\|_X < \infty \right\}.$$

It is clear that $\Theta_{\alpha}^{1}(\mathbb{Z},X)$ is a Banach space under the norm $\|f\|_{\Theta_{\alpha}^{1}} := \sum_{n=-\infty}^{\infty} \|n^{\alpha-1} f(n)\|_X$. If $\alpha=1$ then we simply write $\Theta^{1}(\mathbb{Z},X)$. Now, suppose that $0<\alpha\le 1$. Observe that if $f \in \Theta^1(\mathbb{Z}, X)$, then

$$\|f\|_{\Theta^1_\alpha} := \sum_{n=-\infty}^{\infty} \|n^{\alpha-1} f(n)\|_X < \sum_{n=-\infty}^{\infty} \|f(n)\|_X < \infty.$$

Hence $\Theta^1(\mathbb{Z}, X) \subset \Theta^1_\alpha(\mathbb{Z}, X)$ for $0 < \alpha \leq 1$.

Given $f \in \Theta^1_\alpha(\mathbb{Z}, X)$, it was proved in [5] that

$$\Delta_W^{-\alpha} \Delta f(n) = \Delta \Delta_W^{-\alpha} f(n), \ n \in \mathbb{Z}.$$

Therefore, when the fractional difference operators are defined on $\mathbb{Z}$, there will be no difference between Caputo, Riemann-Liouville and Hilfer. Thus, we will only consider the operator Δ_W^α.

Let $0 < \alpha \leq 1$ and A be a closed linear operator with domain $D(A)$ defined on a Banach space X. Initially, we consider the linear fractional difference equation

$$\Delta_W^\alpha u(n) = Au(n+1) + g(n), \quad n \in \mathbb{Z}. \tag{5.1}$$

We recall from [5, Definition 4.1] that a sequence $u \in \Theta^1(\mathbb{Z}, X)$ is called a *strong solution* for equation (5.1) if $u(n) \in D(A)$ for all $n \in \mathbb{Z}$ and u satisfies (5.1).

Definition 5.1.1 ([5]). *Let A be the generator of a discrete (α,α)-resolvent family $\{S_{\alpha,\alpha}(n)\}_{n\in\mathbb{N}_0}$ and $g : \mathbb{Z} \longrightarrow X$. The sequence*

$$u(n) = \sum_{j=-\infty}^{n-1} S_{\alpha,\alpha}(n-1-j) g(j), \quad n \in \mathbb{Z}, \tag{5.2}$$

is called a mild solution for equation (5.1) if $m \to S_{\alpha,\alpha}(m)g(n-m)$ is summable on $\mathbb{N}_0$ for each $n \in \mathbb{Z}$.

Note that if $g \in \Theta^1(\mathbb{Z}, D(A))$ then each mild solution is a strong one, see [5, Theorem 4.2].

In the following theorem, we establish the existence of (N,λ)-periodic mild solutions for equation (5.1).

Theorem 5.1.2. *Let $0 < \alpha \leq 1$. Assume that A be a closed linear operator defined on a Banach space X, $1 \in \rho(A)$ and*

$$\|(I-A)^{-1}\| < 1.$$

If $g \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$, then there is an (N,λ)-periodic mild solution of (5.1) given by the sequence

$$u(n) := \sum_{j=-\infty}^{n-1} S_{\alpha,\alpha}(n-1-j)g(j), \quad n \in \mathbb{Z}, \tag{5.3}$$

where $\{S_{\alpha,\alpha}(n)\}_{n\in\mathbb{N}_0}$ is discrete (α,α)-resolvent sequence defined in (3.3).

Proof. By Theorem 3.1.7, A generates a summable discrete (α,α)-resolvent sequence $\{S_{\alpha,\alpha}(n)\}_{n\in\mathbb{N}_0}$ given by

$$S_{\alpha,\alpha}(n)x = \sum_{j=0}^{\infty} \varphi_{\alpha,0}(n,j)(I-A)^{-(j+1)}x, \quad n \in \mathbb{N}_0,\ x \in X.$$

Since g is bounded and $\{S_{\alpha,\alpha}(n)\}_{n\in\mathbb{N}_0}$ is summable, it follows that the sequence u is a mild solution of (5.1). It remains to prove that $u \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$. Indeed,

$$\begin{aligned} u(n+N) &= \sum_{j=-\infty}^{n+N-1} S_{\alpha,\alpha}(n+N-1-j)g(j) = \sum_{p=-\infty}^{n-1} S_{\alpha,\alpha}(n-1-p)g(p+N) \\ &= \lambda \sum_{p=-\infty}^{n-1} S_{\alpha,\alpha}(n-1-p)g(p) = \lambda u(n), \end{aligned}$$

getting that $u \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$. □

5.2 Semilinear fractional difference equations

Now, we consider the following fractional difference equation

$$\Delta_W^\alpha u(n) = Au(n+1) + f(n, u(n)), \quad n \in \mathbb{Z}, \tag{5.4}$$

where $0 < \alpha \leq 1$, A satisfies the hypotheses in Theorem 3.1.7 and f satisfies suitable conditions.

Inspired in the solution of the linear case, we give the following definition of mild solution for the semilinear case.

Definition 5.2.1. *Let A be the generator of a discrete (α, α)-resolvent family $\{S_{\alpha,\alpha}(n)\}_{n\in\mathbb{N}_0}$ and $f : \mathbb{Z} \times X \longrightarrow X$. We say that a sequence $u : \mathbb{Z} \longrightarrow X$ is a (N, λ)-periodic mild solution of (5.4) if $u \in P_{N\lambda}(\mathbb{Z}, X)$ and satisfies*

$$u(n) = \sum_{j=-\infty}^{n-1} S_{\alpha,\alpha}(n-1-j)f(j, u(j)), \quad n \in \mathbb{Z}, \tag{5.5}$$

where $m \to S_{\alpha,\alpha}(m)f(n-m, x)$ is summable on $\mathbb{N}_0$ for each $n \in \mathbb{Z}$.

Let $\mathbb{D} := \mathbb{D}(0, 1) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. The following is our main result.

Theorem 5.2.2. *Let $f : \mathbb{Z}\times X \to X$ be given and let A be a closed linear operator defined on a Banach space X such that $1 \in \rho(A)$ and*

$$r_A := \|(I - A)^{-1}\| < 1. \tag{5.6}$$

Assume the following conditions:

- H_1. *There exists $(N, \lambda) \in \mathbb{N} \times (\mathbb{C} \setminus \mathbb{D})$ such that $f(n + N, \lambda x) = \lambda f(n, x)$ for all $(n, x) \in \mathbb{Z} \times X$.*
- H_2. *There exists a constant $L > 0$ such that*

  $$\| f(n, x) - f(n, y) \|_X \leq L \| x - y \|_X,$$

  for all $x, y \in X$ and all $n \in \mathbb{Z}$.
- H_3. *The constant L in H_2 is such that*

  $$L < \left(1 - \frac{1}{|\lambda|^{1/N}}\right)^{\alpha} + \left(\frac{1}{r_A} - 1\right).$$

Then, equation (5.4) has a unique (N, λ)-periodic mild solution.

Proof. First, let us define the operator $G : \mathbb{P}_{N\lambda}(\mathbb{Z},X) \to \mathbb{P}_{N\lambda}(\mathbb{Z},X)$ by

$$G(u)(n) := \sum_{j=-\infty}^{n-1} S_{\alpha,\alpha}(n-1-j)f(j,u(j)).$$

Let $u \in \mathbb{P}_{N\lambda}(\mathbb{Z},X)$ and $g(n) := f(n,u(n))$. By H_1 and Theorem 4.1.10 we get that $g \in P_{N,\lambda}(\mathbb{Z},X)$. As in the linear case, we can see that $G(u) \in \mathbb{P}_{N\lambda}(\mathbb{Z},X)$. It follows that G is well defined. Now, for $u,v \in \mathbb{P}_{N\lambda}(\mathbb{Z},X)$,

$$\| G(u)-G(v) \|_{N\lambda} = \max_{n\in[0,N]\cap\mathbb{Z}} \| \lambda^\wedge(-n) \sum_{j=-\infty}^{n-1} S_{\alpha,\alpha}(n-1-j)\,[f(j,u(j)-f(j,v(j))]\,\|_X,$$

where we have by H_2 that

$$\begin{aligned}
&\| \lambda^\wedge(-(n-1)) \sum_{j=-\infty}^{n-1} S_{\alpha,\alpha}(n-1-j)\,[f(j,u(j)-f(j,v(j))]\,\|_X \\
&=\| \sum_{j=-\infty}^{n-1} \lambda^\wedge(-(n-1-j))S_{\alpha,\alpha}(n-1-j)\lambda^\wedge(-j)\,[f(j,u(j)-f(j,v(j))]\,\|_X \\
&< \sum_{j=-\infty}^{n-1} |\lambda|^\wedge(-(n-1-j))|S_{\alpha,\alpha}(n-1-j)||\lambda|^\wedge(-j)\,\|\,[f(j,u(j)-f(j,v(j))]\,\|_X \\
&< L\sum_{j=-\infty}^{n-1} |\lambda|^\wedge(-(n-1-j))|S_{\alpha,\alpha}(n-1-j)|\,\|\,\lambda^\wedge(-j)\,[u(j)-v(j)]\,\|_X \\
&<\| u-v \|_{N\lambda}\, L\sum_{k=0}^{\infty} \|S^{\sim}_{\alpha,\alpha}(k)\|,
\end{aligned}$$

where $S^{\sim}_{\alpha,\alpha}(n) := \lambda^\wedge(-n)S_{\alpha,\alpha}(n)$. Then,

$$\begin{aligned}
\| G(u)-G(v) \|_{N\lambda} &= \max_{n\in[0,N]\cap\mathbb{Z}} \| \lambda^\wedge(-n) \sum_{j=-\infty}^{n-1} S_{\alpha,\alpha}(n-1-j)\,[f(j,u(j)-f(j,v(j))]\,\|_X \\
&\le L\,\| u-v \|_{N\lambda} \sum_{k=0}^{\infty} \|S^{\sim}_{\alpha,\alpha}(k)\| <\| u-v \|_{N\lambda},
\end{aligned}$$

where by Theorem 3.1.5, Proposition 2.1.7-(iv) and (1.21) we have

$$\begin{aligned}
\sum_{k=0}^{\infty} \|S^{\sim}_{\alpha,\alpha}(k)\| &\le \sum_{n=0}^{\infty} |\lambda|^{-n/N} \sum_{j=0}^{\infty} \varphi_{\alpha,0}(n,j) r_A^{j+1} = \sum_{n=0}^{\infty} |\lambda|^{-n/N}\mathcal{E}_{\alpha,\alpha}\left(1-\frac{1}{r_A},n\right) \\
&= \frac{|\lambda|^{\alpha/N}}{(|\lambda|^{1/N}-1)^\alpha-(1-\frac{1}{r_A})|\lambda|^{\alpha/N}} = \frac{1}{\left(1-\frac{1}{|\lambda|^{1/N}}\right)^\alpha + \left(\frac{1}{r_A}-1\right)}.
\end{aligned}$$

Therefore, the conclusion follows from H_3. From the above, it follows by Banach fixed point theorem that there exists a unique function $u \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$ such that $Gu = u$. Hence u is the unique (N,λ)-periodic mild solution of equation (5.4). $\square$

Remark 5.2.3. *Regarding condition H_3 we observe that is enough to have the weaker condition $L \parallel S^{\sim}_{\alpha,\alpha} \parallel_1 < 1$ where $S^{\sim}_{\alpha,\alpha}(n) := \lambda\hat{}(-n)S_{\alpha,\alpha}(n)$ and $\{S_{\alpha,\alpha}(n)\}_{n\in\mathbb{N}_0}$ is the (α,α)-resolvent sequence generated by A.*

Finally, we finish with an application of the main result presented in this section.

Example 5.2.4. *Let $0 < \alpha < 1$ and $|\lambda| \geq 1$. We consider the following fractional difference-differential equation in $X = L^2(0,1)$*

$$\begin{cases} \Delta^{\alpha}_{W} u(n,x) = \frac{\partial^2}{\partial x^2} u(n+1,x) + g(n,x)\cos(h(n,x)u(n,x)), \quad n \in \mathbb{Z},\ x \in (0,1), \\ \\ u(n,0) = u(n,1) = 0, \end{cases} \tag{5.7}$$

where $g \in \mathbb{P}_{N\lambda}(\mathbb{Z}, L^2(0,1))$, $h \in \mathbb{P}_{N\frac{1}{\lambda}}(\mathbb{Z}, L^2(0,1))$ and

$$\max_{n\in[0,N]\cap\mathbb{Z}} \|g(n)h(n)\|_{L^2} < \left(1 - |\lambda|^{-1/N}\right)^{\alpha} + \left(\left(\sum_{m=1}^{\infty} \frac{1}{(1+(m\pi)^2)^2}\right)^{-1/2} - 1\right). \tag{5.8}$$

We define

$$D(A) = \{f \in L^2(0,1) : f'' \in L^2(0,1), f(0) = f(1) = 0\},$$
$$Af = f'', \quad \forall f \in D(A).$$

Then (5.7) can be written in the abstract setting (5.4). It is well known that A is the generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ on $L^2(0,1)$ (see [61]) which is given by

$$T(t)f = \sum_{j=0}^{\infty} e^{-j^2\pi^2 t}\langle f, \mathbf{e}_j\rangle \mathbf{e}_j.$$

where $\{\mathbf{e}_j\}$ *is the standard basis in* $L^2(0,1)$. *Moreover, we can represent the generator* A *as*

$$Af = -\sum_{m=1}^{\infty}(m\pi)^2\langle f, \mathbf{e}_m\rangle \mathbf{e}_m, \qquad f \in D(A).$$

Then, for each $f \in L^2(0,1)$ *we have* $1 \in \rho(A)$ *and*

$$\|(I-A)^{-1}f\|_{L^2}^2 = \sum_{m=1}^{\infty}\frac{1}{(1+(m\pi)^2)^2}|\langle f, \mathbf{e}_m\rangle|^2.$$

Note that

$$\begin{aligned} r_A := \sup_{\|f\|=1}\|(I-A)^{-1}f\| &= \left(\sum_{m=1}^{\infty}\frac{1}{(1+(m\pi)^2)^2}\right)^{1/2} \\ &\leq \left(\frac{1}{\pi^4}\sum_{m=1}^{\infty}\frac{1}{m^4}\right)^{1/2} = \frac{1}{\sqrt{90}} < 1, \end{aligned}$$

where we have used the formula [70, Pag. 651] in the last equality. Then, the condition (5.6) *is satisfied. Now, we shall verify all the hypotheses in Theorem 5.2.2. Indeed, the sequence* $f(n,\xi) := g(n)\cos(h(n)\xi)$, $\xi \in L^2(0,1)$, *satisfies:*

$$\begin{aligned} f(n+N,\lambda\xi) = g(n+N)\cos(h(n+N)\lambda\xi) &= \lambda g(n)\cos\left(\frac{1}{\lambda}h(n)\lambda\xi\right) \\ &= \lambda g(n)\cos(h(n)\xi) = \lambda f(n,\xi), \end{aligned}$$

and

$$\|f(n,\xi)-f(n,\psi)\|_{L^2} \leq \|g(n)h(n)\|_{L^2}\|\xi-\psi\|_2 \leq L\|\xi-\psi\|_{L^2},$$

where

$$L := \max_{n\in[0,N]\cap\mathbb{Z}}\|g(n)h(n)\|_{L^2}.$$

From equation (5.8) *and the fact that* $r_A < 1$, *we obtain that*

$$L < \left(1-\frac{1}{|\lambda|^{1/N}}\right)^{\alpha} + \left(\frac{1}{r_A}-1\right),$$

satisfying H_3. *Thus, we have checked all the hypotheses of Theorem 5.2.2. Hence equation* (5.7) *has a unique* (N,λ)*-periodic mild solution.*

Finally, observe that in case $|\lambda| = 1$, *we have that*

$$L := \max_{n \in [0,N] \cap \mathbb{Z}} \|g(n)h(n)\|_{L^2} < \frac{1}{r_A} - 1.$$

and therefore condition H_3 *independent of* α. *This happens precisely in the standard cases of discrete periodic, discrete anti-periodic and discrete Bloch periodic functions.*

Publications

The dissertation is based on the following papers:

[2] L. Abadias, E. Alvarez and S. Díaz, Subordination principle, Wright functions and large-time behavior for the discrete in time fractional diffusion equation, *J. Math. Anal. Appl.*, 507(1),125741, 2022.

[15] E. Alvarez, S. Díaz, and C. Lizama, On the existence and uniqueness of (N, λ)-periodic solutions to a class of Volterra difference equations, *Adv. Difference Equ.*,(105):1-12, 2019.

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